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Transposition in Quantum Information Theory

Masterarbeit von Alexander Müller-Hermes

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Ich erkläre hiermit, dass ich diese Masterarbeit selbständig und nur mit den angegebenen Hilfsmitteln angefertigt habe.

(Garching b. München den 10.09.2012, Alexander Müller-Hermes)

Abstract

In this thesis we study applications of the transposition map in quantum information theory. We are interested in particular in the problem of detecting whether a given quantum state is distillable and in the problem of bounding the quantum capacity of a quantum channel. In these cases there are well-known results involving the transposition map. Our main goal is to characterize different maps, which would lead to similar criteria or bounds. The existence of such maps would provide useful tools for these applications. We show, by using the theory of linear preservers and convex analysis, that in many cases the transposition up to some transformation is the only map that can give such results.

Zusammenfassung

Wir untersuchen Anwendungen der Transpositions-Abbildung in der Quanteninformationstheorie. Dabei interessieren wir uns speziell für das Problem Distillierbarkeit eines gegebenen Quantenzustands zu erkennen und für das Problem Schranken an die Quantenkapazität von Quantenkanälen zu finden. In diesen Fällen gibt es bereits Ergebnisse, die auf den Eigenschaften der Transposition beruhen, und es ist unser Ziel weitere Abbildungen zu charakterisieren, die zu ähnlichen Kriterien für Nichtdistillierbarkeit und zu ähnlichen Schranken führen würden. Die Existenz solcher Abbildungen würde nützliche Werkzeuge für diese Anwendungen geben. Wir zeigen mithilfe der Theorie Linearer Erhaltungsprobleme und konvexer Analysis, dass bis auf bestimmte Transformationen die Transposition die einzige Abbildung ist, die solche Resultate ermöglicht.

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Contents

1. Basic Definitions and Terminology

In the field of information theory one tries to understand the properties of information processing in physical systems in a rigorous mathematical way. Theorems such as Shannon's Theorem [Sha48] give a mathematical precise meaning to what useful measures of information are, and how much information can be transferred from one system to another using a physical process. As information theory is not just dealing with abstract mathematical quantities, but has the premise to give results for physical systems, it is influenced by the physical laws that we use to describe such systems.

Quantum information theory is a generalization of the classical theory, where the underlying physics works by the laws of quantum mechanics. It can be shown to be more powerful than the classical theory in many cases, see [MN07], and thus is a proper generalization of standard information theory. There are also more difficulties, which have to be handled in the quantum case. The various kinds of capacities for quantum channels [Llo97] [Dev05] [Wil12] are one example. They are defined, in order to describe different families of protocols assisted by different resources.

In the present chapter we will introduce the basic concepts and tools of quantum information theory, which we will use later in our argumentation.

1.1. Quantum States and Classes of Matrix Maps

In this section we will present the basic definitions and the terminology of quantum theory. Most of this will be known to anyone working in the field of quantum information theory and can be skipped if necessary. For more details on the formalism of quantum mechanics and quantum information theory see [MN07].

In order to present the basic setting, we will start with the definition of a quantum state.

Definition 1. (*Quantum State*)

Let $\mathfrak{h}$ be a finite-dimensional, complex Hilbert space. A ***quantum state*** *is an operator $X \in \mathfrak{B}(\mathfrak{h})$ fulfilling $X \geq 0$ and $tr(X) = 1$.*

Most of the time we will identify an n-dimensional complex Hilbert spaces $\mathfrak{h}$ with the standard complex Hilbert space $\mathbb{C}^n$. For vectors we will use the bra-ket-notation and denote column vectors as $|u\rangle \in \mathbb{C}^n$ and conjugated row vectors as $\langle u| \in (\mathbb{C}^n)'$. With this in mind we write the standard scalar product on $\mathbb{C}^n$ as $\langle u|v\rangle$, where the first vector is conjugated, as it should be.

The operator algebra $\mathfrak{B}(\mathfrak{h})$ of bounded operators on $\mathfrak{h}$ will be often identified with the matrix algebra $\mathfrak{M}_n$ of complex $n \times n-$matrices, which is also a Hilbert space, with the Hilbert-Schmidt inner product $\langle X, Y\rangle = \operatorname{tr}\left(X^\dagger Y\right)$. We define $\mathfrak{H}_n \subset \mathfrak{M}_n$ to be the subspace of hermitian $n \times n-$matrices, $\mathfrak{M}_n^+ \subset \mathfrak{M}_n$ to be the subset of positive $n \times n-$matrices and $\mathfrak{U}_n \subset \mathfrak{M}_n$ to be the subset of unitary $n \times n-$matrices.

As a positive operator a quantum state is necessarily hermitian and therefore admits a spectral decomposition, which can be used to give an interpretation to the above definition. There are $p_i \in \mathbb{R}^+$ for all $i \in \{1, \dots, n\}$ fulfilling $\sum_i p_i = 1$ and pairwise orthogonal projectors $|i\rangle\langle i| \in \mathfrak{M}_n$ fulfilling $\sum_i |i\rangle\langle i| = I_n$ such that

$$X = \sum_{i=1}^{n} p_i |i\rangle\langle i| \ .$$

Projectors like $|i\rangle\langle i| \in \mathfrak{M}_n$ are also called **pure states** and they correspond to vectors $|i\rangle \in \mathbb{C}^n$ in the underlying Hilbert space. These states are used to describe systems, whose state is exactly known. But as quantum theory is an inherent statistical theory, it is useful to allow for statistical mixtures of pure states.

The above decomposition of the state $X \in \mathfrak{M}_n$ shows that X can be interpreted as a statistical mixture of the pure states $|i\rangle\langle i| \in \mathfrak{M}_n$ with probabilities $p_i \in \mathbb{R}^+$, which is why such a state is also called a **mixed state**. The normalization of the trace corresponds to normalization of the probability distribution $(p_i)_i$. Note that the decomposition of a quantum state X into a statistical mixture of pure states is in general not unique. There may be different possibilities to obtain the state from different probability distributions and pure states, which are not pairwise orthogonal.

Remark 1. *We will not go into further details of how to actually model a quantum system occurring in real world applications. In the following we will look at quantum theory from an abstract point of view involving only quantum states and certain maps, that can be applied to them. It might not even be possible to implement some of these maps physically. For details on the physics and for further explanations of the formalism see [MN07].*

After introducing quantum states we want to define physical processes in this formalism. We will introduce them in the most general way possible, namely as maps on the spaces of bounded operators with the only constraint that quantum states should be mapped to quantum states.

Before we introduce these maps, we will state some further notational conventions.

Remark 2.
1. *When not said otherwise we will denote complex vectors $|u\rangle \in \mathbb{C}^n$ by small Latin letters and complex matrices $X \in \mathfrak{M}_n$ by capital Latin letters. Maps $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ between matrix spaces are denoted by capital calligraphic Latin letters.*
2. *In quantum information theory we will often work in bipartite settings, where the Hilbert space $\mathfrak{h}$ is a tensor product of two "local" Hilbert spaces. These local Hilbert spaces correspond to different parties, which can perform operations only on their own Hilbert space, e.g. because of spatial distance. We will adopt the historical notation and call one party "Alice" (A) and the other party "Bob"(B) and write $\mathfrak{h} = \mathfrak{h}_A \otimes \mathfrak{h}_B$, or $\mathbb{C}^n = \mathbb{C}^{n_A} \otimes \mathbb{C}^{n_B}$.*

For the operator algebras we would write $\mathfrak{B}(\mathfrak{h}_A \otimes \mathfrak{h}_B)$ *or* $\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}$ *or* $\mathfrak{M}_{n_A n_B}$. *Note that the latter does* ***not*** *mean* $n_A \times n_B-$*matrices, which we denote by* $\mathfrak{M}_{n_A, n_B}$.

3. *There is one exception from the above notation. On the matrix algebra* $\mathfrak{M}_n$, *we can define the well-known n-dimensional transposition map, which we denote by* $\vartheta_n : \mathfrak{M}_n \to \mathfrak{M}_n$. *Note that by using this map we fix a basis of the matrix space, namely the basis with respect to which the transposition is defined in the usual way. When we apply the transposition to a concrete matrix, we will also write* X^T. *Sometimes we apply the transposition partially, which means applying the map* $id \otimes \vartheta_B : \mathfrak{M}_{n_A n_B} \to \mathfrak{M}_{n_A n_B}$. *We also write* X^{T_B} *for* $X \in \mathfrak{M}_{n_A n_B}$.

4. *On the bipartite Hilbert space* $\mathfrak{h} = \mathbb{C}^n \otimes \mathbb{C}^n$ *we can define the vector*

$$|\Omega_n\rangle = \sum_{i=1}^{n} |ii\rangle$$

with respect to a fixed orthonormal basis $\{|i\rangle\}_i \subset \mathbb{C}^n$. *This vector is called the* ***maximally entangled vector***. *Note that this vector is not normalized. The pure state corresponding to this vector is the projector* $\frac{\omega_n}{n} := \frac{|\Omega_n\rangle\langle\Omega_n|}{n} \in \mathfrak{M}_n \otimes \mathfrak{M}_n$ *and we will call this state the* ***maximally entangled state***. *By applying the partial transposition to the maximally entangled state, we obtain the so called Flip-operator* $\mathbb{F}_n = (id \otimes \vartheta_n)(\omega_n)\, \mathfrak{M}_n \otimes \mathfrak{M}_n$. *This operator acts on product-vectors as* $\mathbb{F}_n |ij\rangle = |ji\rangle$.

Now we can introduce the necessary terminology to define physical maps between the matrix spaces.

Definition 2. (*Properties of Matrix Maps*)
A linear map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *is called*

- ***positive*** *if* $\mathcal{T}(X) \in \mathfrak{M}_m^+$ *for all* $X \in \mathfrak{M}_n^+$.
- ***completely positive*** *if* $(id_k \otimes \mathcal{T})$ *is positive for all* $k \in \mathbb{N}$.
- ***trace-preserving*** *if* $tr(\mathcal{T}(X)) = tr(X)$ *for all* $X \in \mathfrak{M}_n$.
- ***unital*** *if* $\mathcal{T}(I) = I$.

Example 1. 1. *The transposition* $\vartheta_n : \mathfrak{M}_n \to \mathfrak{M}_n$ *is positive, trace-preserving, unital but not completely positive, as*

$$(id_n \otimes \vartheta_n)(\omega_n) = \mathbb{F}_n \ngeq 0$$

2. *The map* $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *with* $\mathcal{T}(X) = tr(X)\, A$ *for a matrix* $A \in \mathfrak{M}_m$ *is completely positive, but not necessarily trace-preserving or unital.*

With this terminology we define the physical maps, also called quantum channels. Note that a quantum state corresponds to a positive matrix with trace 1.

Definition 3. (*Quantum Channel*)
A map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ which is completely positive and trace-preserving is called a ***Quantum Channel****.*

It is clear by the definition of a quantum state, that $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ has to be trace-preserving, in order to preserve the trace of quantum states.

The completely positivity might not be that clear on first sight. It is not enough to assume $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ to be positive and trace-preserving. We cannot define a matrix space of the whole universe, whatever this is. Therefore we would like to have a local description of physical processes and thus define maps that act on local matrix spaces. But if the maps are defined locally, we can construct a new map, by combining the local system with an environment. For an arbitrary $k \in \mathbb{N}$ we define the map $\mathrm{id}_k \otimes \mathcal{T} : \mathfrak{M}_k \otimes \mathfrak{M}_n \to \mathfrak{M}_k \otimes \mathfrak{M}_n$. This map acts trivially on the k-dimensional environment $\mathfrak{M}_k$ and applies the quantum channel to the local system. As this construction leads to a new physical process it has to map quantum states to quantum states, if the formalism represents reality in a consistent way. But this means that $\mathrm{id}_k \otimes \mathcal{T} : \mathfrak{M}_k \otimes \mathfrak{M}_n \to \mathfrak{M}_k \otimes \mathfrak{M}_n$ has to be positive for all $k \in \mathbb{N}$ and thus $\mathcal{T}$ completely positive.

What we introduced so far is called the Schrödinger picture, where a quantum channel is a trace-preserving completely positive map acting on quantum states. By taking adjoints with respect to the Hilbert-Schmidt inner product, we arrive at an equivalent picture, which is called the Heisenberg picture, where a quantum channel acts on operators defining measurements. In this picture a quantum channel is a unital completely positive map. We will mostly use Schrödinger picture and note, when we are doing calculations in the Heisenberg-picture.

In the following we will need some more classes of matrix maps, which contain also unphysical maps.

Definition 4. (*Properties of Matrix Maps*)
A linear map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ is called

- ***hermiticity preserving*** *if $\mathcal{T}(X) \in \mathfrak{H}_n$ for all $X \in \mathfrak{H}_n$.*
- ***k-positive*** *if $id_k \otimes \mathcal{T}$ is positive.*
- ***completely co-positive*** *if ϑT is completely positive.*
- ***decomposable*** *if $\mathcal{T} = \mathcal{T}_1 + \vartheta\mathcal{T}_2$ for $\mathcal{T}_1, \mathcal{T}_2$ completely positive.*

The importance of most of these sets will become apparent in the later argumentation. The class of decomposable maps deserves some separate comment. This class of positive maps is of interest, because of the simple structure of its elements. There was the conjecture, whether all positive maps have this structure, which has been falsified by Woronowicz in [Wor76].

Remark 3. *We will denote the sets introduced so far in the following way.*

- $\mathfrak{HP}(\mathfrak{M}_n, \mathfrak{M}_m) := \{\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *hermiticity preserving*$\}$.
- $\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m) := \{\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *positive*$\}$.
- $\mathfrak{P}_k(\mathfrak{M}_n, \mathfrak{M}_m) := \{\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *k-positive*$\}$.
- $\mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_m) := \{\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *completely positive*$\}$.
- $\mathfrak{coCP}(\mathfrak{M}_n, \mathfrak{M}_m) := \{\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *completely co-positive*$\}$.
- $\mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_m) \vee \mathfrak{coCP}(\mathfrak{M}_n, \mathfrak{M}_m) := \{\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *decomposable*$\}$.
- $\mathfrak{CPTP}(\mathfrak{M}_n, \mathfrak{M}_m) := \{\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *completely positive and trace preserving*$\}$.
- $\mathfrak{CPU}(\mathfrak{M}_n, \mathfrak{M}_m) := \{\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *completely positive and unital*$\}$.

In the following part we will introduce the Choi-Jamiolkowski-isomorphism, which is an isomorphism between $\mathfrak{B}(\mathfrak{M}_n, \mathfrak{M}_m)$ and $\mathfrak{M}_{nm}$. This isomorphism will map every matrix map to a matrix, the so called Choi-matrix. Certain properties of the matrix map will correspond to properties of this matrix, which are in many cases easier to handle.

Before introducing the isomorphism we need the well-known Schmidt-decomposition, which is widely used in quantum information theory.

Lemma 1. (*Schmidt-decomposition, Schmidt-rank, see* [*MN07*])

Let $|u_{AB}\rangle \in \mathbb{C}^{n_A} \otimes \mathbb{C}^{n_B}$ *be a vector in a bipartite Hilbert space, then there exist orthonormal bases* $\{|k_A\rangle\}_{k=1}^{n_A} \subset \mathbb{C}^{n_A}$, $\{|k_B\rangle\}_{k=1}^{n_B} \subset \mathbb{C}^{n_B}$ *and* $\{s_k\}_{k=1}^{\min(n_A,n_B)} \subset \mathbb{R}_0^+$ *such that*

$$|u_{AB}\rangle = \sum_{k=1}^{\min(n_A,n_B)} s_k |k_A\rangle \otimes |k_B\rangle \ .$$

The $(s_k)_{k=1}^{\min(n_A,n_B)} \subset \mathbb{R}_0^+$ *ordered in decreasing order are called* ***Schmidt-coefficients*** *of* $|u_{AB}\rangle \in \mathbb{C}^{n_A} \otimes \mathbb{C}^{n_B}$ *with respect to the partition (A:B).*

The number $\mathcal{SR}(|u_{AB}\rangle) := \max(k \in \mathbb{N} : s_k \neq 0) \leq \min(n_A, n_B)$ *is called* ***Schmidt-rank*** *of* $|u_{AB}\rangle \in \mathbb{C}^{n_A} \otimes \mathbb{C}^{n_B}$ *with respect to the partition (A:B).*

Proof. We can write $|u_{AB}\rangle \in \mathbb{C}^{n_A} \otimes \mathbb{C}^{n_B}$ in an arbitrary orthonormal product basis as

$$|u_{AB}\rangle = \sum_{i,j} c_{ij} |i\rangle \otimes |j\rangle \ .$$

A singular value decomposition of the matrix $(C)_{i,j} = c_{ij}$, with unitaries $U \in \mathfrak{U}_{n_A}$ and $V \in \mathfrak{U}_{n_B}$ and a rectangular diagonal matrix $S \in \mathfrak{M}_{n_A,n_B}$ with diagonal elements $(s_k)_k$ ordered in decreasing order, gives

$$
\begin{aligned}
|u_{AB}\rangle &= \sum_{i,j} c_{ij} \, |i\rangle \otimes |j\rangle \\
&= \sum_{i,j,k} u_{i,k} s_k v_{k,j} \, |i\rangle \otimes |j\rangle \\
&= \sum_k s_k \left(\sum_i u_{i,k} \, |i\rangle \right) \otimes \left(\sum_j v_{k,j} \, |j\rangle \right) \\
&= \sum_k s_k \, |k_A\rangle \otimes |k_B\rangle \ ,
\end{aligned}
$$

where we used that a unitary transformation of an orthonormal basis gives again an orthonormal basis.

□

The Schmidt-Rank can be seen as a measure of entanglement for pure bipartite states. A low Schmidt-Rank allows , via the above decomposition, a more efficient representation of the state, because the basis vectors corresponding to zero Schmidt-coefficients can be discarded.

Another way of viewing the Schmidt-decomposition is the following:

Corollary 1. *Let $|u_{AB}\rangle \in \mathbb{C}^{n_A} \otimes \mathbb{C}^{n_B}$ be a vector in a bipartite Hilbert space, then there exists a matrix $A \in \mathfrak{M}_{n_A,n_B}$, with rank$(A) = \mathcal{SR}\left(|u_{AB}\rangle\right)$ such that*

$$
\begin{aligned}
|u_{AB}\rangle &= \left(I_{n_A} \otimes A\right) |\Omega_{n_A}\rangle \\
&= \left(A^T \otimes I_{n_B}\right) |\Omega_{n_B}\rangle \ .
\end{aligned}
$$

Proof. Take the Schmidt-decomposition

$$
|u_{AB}\rangle = \sum_{k=1}^{\min(n_A,n_B)} s_k \, |k_A\rangle \otimes |k_B\rangle \ .
$$

We can choose $A = \sum_k s_k \, |k_B\rangle \langle k_A|$ and a simple calculation shows the corollary.

□

We are now in the position to introduce the Choi-Jamiolkowski isomorphism. In the following we will fix an orthonormal basis $\{|i\rangle\}$ of $\mathbb{C}_n$.

Definition 5. (*Choi-Matrix*)
Let $\omega_n = |\Omega_n\rangle\langle\Omega_n|$ *for* $|\Omega_n\rangle = \sum_{i=1}^n |ii\rangle \in \mathbb{C}_n \otimes \mathbb{C}_n$. *For a linear map* $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *the matrix*

$$C_{\mathcal{T}} = id_n \otimes \mathcal{T}(\omega_n) \in \mathfrak{M}_{nm}$$

is called the ***Choi-Matrix*** *corresponding to* $\mathcal{T}$.

By expanding the maximally entangled state, we get

$$C_{\mathcal{T}} = \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes \mathcal{T}(|i\rangle\langle j|)$$

for the Choi-matrix of $\mathcal{T}$. Therefore the Choi-matrix is a block matrix, with $\mathcal{T}(|i\rangle\langle j|)$ in the block corresponding to $|i\rangle\langle j|$. As these rank-1 matrices form a basis of $\mathfrak{M}_n$, $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ is uniquely determined by the matrices $(\mathcal{T}(|i\rangle\langle j|))_{i,j}$. This shows that the map $\mathcal{T} \mapsto C_{\mathcal{T}}$ is a bijection. This is also part of the following theorem.

Theorem 1. (*Choi-Jamiolkowski Isomorphism*)
For a linear map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *the unique solution* $C \in \mathfrak{M}_{nm}$ *of*

$$tr(\mathcal{T}(A)B) = tr\left(CA^T \otimes B\right) \ \forall A \in \mathfrak{M}_n \forall B \in \mathfrak{M}_m$$

is given by $C = C_{\mathcal{T}}$, *where* $C_{\mathcal{T}}$ *is taken with respect to the basis used for* $\mathfrak{M}_n$. *Furthermore the map*

$$\mathcal{T} \mapsto C_{\mathcal{T}}$$

is an isomorphism between the spaces $\mathfrak{L}(\mathfrak{M}_n, \mathfrak{M}_m)$ *and* $\mathfrak{M}_{nm}$, *called the* ***Choi-Jamiolkowski Isomorphism***.

Proof. We have already seen, that the map $\mathcal{T} \mapsto C_{\mathcal{T}}$ is a bijection. What remains is the following explicit calculation

$$\begin{aligned}
\operatorname{tr}\left(CA^T \otimes B\right) &= \sum_{ij} \operatorname{tr}\left(|i\rangle\langle j| A^T \otimes \mathcal{T}(|i\rangle\langle j|) B\right) \\
&= \sum_{ij} \langle j| A^T |i\rangle \operatorname{tr}(\mathcal{T}(|i\rangle\langle j|) B) \\
&= \operatorname{tr}\left(\mathcal{T}\left(\sum_{ij} \langle j| A^T |i\rangle |i\rangle\langle j|\right) B\right) \\
&= \operatorname{tr}(\mathcal{T}(A) B) \ .
\end{aligned}$$

It is clear that $C_{\mathcal{T}}$ is the only solution of the equation.

□

With the use of the Choi-Jamiolkowski Isomorphism it is possible to characterize many classes of maps between matrix algebras.

Corollary 2. (*Characterization via Choi-Matrices*)
A linear map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ *is*

1. ***hermiticity preserving*** *iff* $C_{\mathcal{T}} \in \mathfrak{H}_{nm}$.
2. ***positive*** *iff* $\langle u| \otimes \langle v| C_{\mathcal{T}} |u\rangle \otimes |v\rangle \geq 0$ *for all* $|u\rangle \in \mathbb{C}_n, |v\rangle \in \mathbb{C}_m$.
3. ***k-positive*** *iff* $\langle u| C_{\mathcal{T}} |u\rangle \geq 0$ *for all* $|u\rangle \in \mathbb{C}_{nm}$ *with* $\mathcal{SR}(|u\rangle) \leq k$.
4. ***completely positive*** *iff* $C_{\mathcal{T}} \in \mathfrak{M}^+_{nm}$.
5. ***completely co-positive*** *iff* $(id \otimes \vartheta)(C_{\mathcal{T}}) \in \mathfrak{M}^+_{nm}$.
6. ***trace-preserving*** *iff* $tr_B(C_{\mathcal{T}}) = I_n$.
7. ***unital*** *iff* $tr_A(C_{\mathcal{T}}) = I_m$.

Proof. The proofs are straightforward:

1. If $\mathcal{T} \in \mathfrak{HP}(\mathfrak{M}_n, \mathfrak{M}_m)$ then we get $C^\dagger_{\mathcal{T}} = \sum_{i,j} |i\rangle\langle j|^\dagger \otimes \mathcal{T}(|i\rangle\langle j|)^\dagger = \sum_{i,j} |j\rangle\langle i| \otimes \mathcal{T}(|j\rangle\langle i|) = C_{\mathcal{T}}$. In the other direction we get that $\mathcal{T}(|i\rangle\langle j|)^\dagger = \mathcal{T}(|j\rangle\langle i|)$ and therefore $\mathcal{T} \in \mathfrak{HP}(\mathfrak{M}_n, \mathfrak{M}_m)$ whenever $C_{\mathcal{T}}$ is hermitian.
2. This follows from: $\langle u| \otimes \langle v| C_{\mathcal{T}} |u\rangle \otimes |v\rangle = \langle v| \mathcal{T}(|u\rangle\langle u|) |v\rangle$ and the definition of positivity.
3. For all $|u\rangle \in \mathbb{C}_{kn}$ and $|v\rangle \in \mathbb{C}_{km}$: $\langle v| (\mathrm{id}_k \otimes \mathcal{T})(|u\rangle\langle u|) |v\rangle = \langle v| \left(U^T \otimes I_m\right) C_{\mathcal{T}} \left(\overline{U} \otimes I_m\right) |v\rangle$ for some $U \in \mathfrak{M}_{k,n}$ with $\mathrm{rank}(U) = k$, where we used Corollary 1. As $\mathcal{SR}\left(\langle v| \left(U^T \otimes I_m\right)\right) \leq k$ and as all vectors with $|v\rangle \in \mathbb{C}_{nm}$ with $\mathcal{SR}(|v\rangle) \leq k$ can be written this way, we are finished.
4. This is a special case of 3. for k=n.
5. Follows from 4. and from the definition of $\mathfrak{coCP}(\mathfrak{M}_n, \mathfrak{M}_m)$.
6. We have: $\mathrm{tr}_B(C_{\mathcal{T}}) = \sum_{i,j} \mathrm{tr}(\mathcal{T}(|i\rangle\langle j|)) |i\rangle\langle j|$. The proof is finished, as $\mathcal{T}$ is trace-preserving iff $\mathrm{tr}(\mathcal{T}(|i\rangle\langle j|)) = \delta_{ij}$.
7. We have: $\mathrm{tr}_A(C_{\mathcal{T}}) = \sum_i \mathcal{T}(|i\rangle\langle i|) = \mathcal{T}(I_n)$. This finishes the proof.

□

The Choi-Jamiolkowski Isomorphism also allows a consistent treatment of the important representation theorems that exist for hermitian preserving maps and the special cases of completely positive maps.

Corollary 3. (*Kraus-representation*)
A linear map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ is

- ***hermiticity preserving*** *iff it has the representation $\mathcal{T}(X) = \sum_{i=1}^{d} \epsilon_i A_i^\dagger X A_i$ with matrices $A_i \in \mathfrak{M}_{n,m}$ and signs $\epsilon_i = \pm 1$.*

- ***completely positive*** *iff it has the representation $\mathcal{T}(X) = \sum_{i=1}^{d} A_i^\dagger X A_i$ with matrices $A_i \in \mathfrak{M}_{n,m}$.*

The matrices $A_i \in \mathfrak{M}_{n,m}$ are called Kraus-operators.

Proof. By the Corollary 1 to the Schmidt-decomposition we can write the rank-1 idempotent corresponding to $|u\rangle \in \mathbb{C}^{nm}$ as

$$|u\rangle\langle u| = (I_n \otimes A)\,\omega_n \left(I_n \otimes A^\dagger\right) ,$$

with a matrix $A \in \mathfrak{M}_{n,m}$.

The corollary follows from a spectral decomposition of the hermitian Choi-Matrix,

$$C_{\mathcal{T}} = \sum_{k=1}^{d} \epsilon_k |e_k|\,|\tilde{u}_k\rangle\langle\tilde{u}_k| = \sum_{k=1}^{d} \epsilon_k\,|u_k\rangle\langle u_k|$$

for $d \le nm$ and with $|u_k\rangle = \sqrt{|e_k|}\,|\tilde{u}_k\rangle$. We denote by $(\epsilon_k|e_k|)_k$ the eigenvalues of $C_{\mathcal{T}}$. From the above representation of rank-1 idempotents we get

$$\begin{aligned} C_{\mathcal{T}} &= \sum_{k=1}^{d} \epsilon_k\,(I_n \otimes A_k)\,\omega_n \left(I_n \otimes A_k^\dagger\right) \\ &= \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes \left(\sum_{k=1}^{d} A_k\,|i\rangle\langle j|\,A_k^\dagger\right) \end{aligned}$$

Comparing this with

$$C_{\mathcal{T}} = \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes \mathcal{T}(|i\rangle\langle j|)$$

and using the Choi-Jamiolkowski Isomorphism leads to the desired representation.

□

As an application of the above decomposition we can introduce the quantum channel corresponding to a quantum measurement. Note that we will only treat projector-valued measures here. For a more general treatment see [MN07].

Definition 6. (*Projector-valued measures*)

*A **projector-valued measure** on $\mathfrak{M}_n$ is a set of projectors $\{P_i\}_{i=1}^k \subset \mathfrak{M}_n$ such that*

$$\sum_{i=1}^{k} P_i = I$$

and $tr(P_i P_j) = \delta_{ij}$. When measuring a quantum state $X \in \mathfrak{M}_n$, we obtain the measurement outcome $i \in \{1, \ldots, k\}$ with probability

$$p_i = tr(XP_i) \ .$$

After the measurement gave the outcome $i \in \{1, \ldots, k\}$, the state of the system is given by

$$X_i = \frac{P_i X P_i}{p_i} \ .$$

By writing the state after the measurement as a statistical mixture of the above states, we can associate the quantum channel $\mathcal{P} : \mathfrak{M}_n \to \mathfrak{M}_n$ defined by

$$\mathcal{P}(X) = \sum_{i=1}^{k} p_i X_i = \sum_{i=1}^{k} P_i X P_i$$

to the projector-valued measure.

There are more general ways to define measurements in quantum theory [MN07], but it will be sufficient for us to use projector-valued measures. As an example of such a measurement, we will introduce the following teleportation protocol:

Example 2. (*Implementation of a quantum channel via Teleportation. [Wol12]*)

For a quantum channel $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ consider the state

$$X \otimes \frac{C_{\mathcal{T}}}{n} \in \mathfrak{M}_n^A \otimes \mathfrak{M}_n^A \otimes \mathfrak{M}_m^B \ ,$$

where we introduced a bipartition (A:B) of the whole system and a local quantum state $X \in \mathfrak{M}_n^A$ at A. The normalized Choi-matrix $\frac{C_{\mathcal{T}}}{n}$, which is a quantum state for a quantum channel $\mathcal{T}$, is shared between the two parties. Now we apply the projector-valued measure

$$\{\frac{\omega_n^A}{n}, I_n^A \otimes I_n^A - \frac{\omega_n^A}{n}\}$$

to the A part of the system.

By a simple calculation, according to Definition 6, we obtain the final state after measuring *the outcome 1 corresponding to* $\frac{\omega_n^A}{n}$*:*

$$X_1 = \frac{\left(\frac{\omega_n^A}{n} \otimes I_m^B\right)\left(X \otimes \frac{C_{\mathcal{T}}}{n}\right)\left(\frac{\omega_n^A}{n} \otimes I_m^B\right)}{p_1} = \frac{1}{n^2}\frac{\frac{\omega_n^A}{n} \otimes T(X)}{p_1}.$$

From normalization of X_1 *we get* $p_1 = \frac{1}{n^2}$*. This means that with a probability of* p_1 *we have applied the quantum channel* $\mathcal{T}$ *to the quantum state* $X \in \mathfrak{M}_n$ *and teleported the result from A to B.*

Note that by applying this protocol to the identity channel, we obtain a standard quantum teleportation protocol.

Remark 4. *The probability* $p_s = \frac{1}{n^2}$ *of successful teleportation in the above protocol can be improved further, as shown in [Wol12], by using a channel-dependent projector-valued measure and allowing for classical communication. Consider a set* $\{U_i\}_{i=1}^k \subset \mathcal{U}_n$ *of unitaries, which are pairwise orthogonal with respect to the Hilbert Schmidt inner product, and for which there exist pairwise orthogonal unitaries* $\{V_i\}_{i=1}^k \subset \mathfrak{U}_m$ *such that* $V_i\mathcal{T}(X)V_i^\dagger = \mathcal{T}\left(U_iXU_i^\dagger\right)$*.*

Apply the projector-valued measure $\{(I \otimes U_i)\,\omega\,(I \otimes U_i)^\dagger\}_{i=1}^k \cup \{P^\perp\}$*, where* $P^\perp$ *denotes the projector on the orthogonal subspace with respect to the other projectors, to the system defined in Example 2. In each of the outcomes corresponding to the operators* $(I_n \otimes U_i)\,\omega_n\,(I_n \otimes U_i)^\dagger$*, we can correct the occurring error by applying the corresponding* V_i *on the B part of the system. This can be done by classical communication as A can communicate the measurement outcome i to B. By computing all the probabilities we obtain a success-probability of* $p_s = \frac{k}{n^2}$*.*

The channels, which can be implemented perfectly via teleportation, i.e. with success probability $p_s = 1$ *are the channels* $\mathcal{T}$ *which preserve unitary equivalence, i.e. for all* $U \in \mathfrak{U}_n$ *exists a* $V \in \mathfrak{U}_m$ *such that* $V\mathcal{T}(X)V^\dagger = \mathcal{T}\left(UXU^\dagger\right)$*. One example of such a channel is the identity channel, which shows, that the standard quantum teleportation protocol can be implemented with probability* $p_s = 1$*.*

We will proof later, that the unitary equivalence preserving channels are of a very particular structure. These channels are also called Werner channels and we will obtain their structure in Theorem 4.

So far we introduced quantum channels and projector-valued measurements as an example for quantum channels. We will use the rest of this section to classify quantum channels with respect to different means, which are justified from different communication tasks.

For a quantum channel $\mathcal{T} \in \mathfrak{CPTP}(\mathfrak{M}_n, \mathfrak{M}_m)$ the map $\mathrm{id}_n \otimes \mathcal{T}$ can be interpreted as the physical process that acts as T on a subsystem but leaves an n-dimensional environment unchanged. When applied to the the maximally entangled state $\frac{\omega_n}{n}$ between the system and the n-dimensional environment we obtain $\frac{C_{\mathcal{T}}}{n}$. Therefore we can use the Choi-Matrix to quantify how well entanglement with an environment is preserved, when the channel is applied.

This gives rise to the following important class of completely positive maps:

Definition 7. (*k-Superpositive Maps*)
A linear map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ is called ***k-superpositive*** *if*

$$C_{\mathcal{T}} \in span^{+}\left(\{|\psi\rangle\langle\psi| : SR(|\psi\rangle) \leq k\}\right) .$$

We will use the notation

$$\mathfrak{SP}_k\left(\mathfrak{M}_n, \mathfrak{M}_m\right) = \{\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m \ k\text{-superpositive}\} .$$

In particular 1-superpositive maps are also called ***entanglement breaking*** *and are the maps for which $C_{\mathcal{T}}$ is separable.*

The entanglement breaking channels are the channels, which completely destroy any entanglement the input state had with an environment. Therefore these channels are not useful in quantum information processing tasks, in which such entanglement has to be preserved.

The maps in $\mathfrak{SP}_k\left(\mathfrak{M}_n, \mathfrak{M}_m\right)$ can also be characterized via their Kraus-decomposition:

Corollary 4. *We have $\mathcal{T} \in \mathfrak{SP}_k\left(\mathfrak{M}_n, \mathfrak{M}_m\right)$ iff it has the representation $\mathcal{T}(X) = \sum_{i=1}^{d} A_i^{\dagger} X A_i$ with matrices $A_i \in \mathfrak{M}_{n,m}$ such that $rank(A_i) \leq k$.*

Proof. This follows immediately from the proof of Corollary 3, where we obtained the Kraus-operators from the representation

$$|u\rangle\langle u| = (I_n \otimes A)\, \omega_n \left(I_n \otimes A^{\dagger}\right) ,$$

of the rank-1-idempotents corresponding to the eigenvectors $|u\rangle \in \mathbb{C}^{nm}$ of the Choi-matrix $C_{\mathcal{T}}$. As these eigenvectors have $\mathcal{SR}(|u\rangle) \leq k$ for $\mathcal{T} \in \mathfrak{SP}_k\left(\mathfrak{M}_n, \mathfrak{M}_m\right)$, we can choose $A \in \mathfrak{M}_{n,m}$ with $\text{rank}(A) \leq k$, according to Corollary 1. The other direction is clear.

□

So far we classified completely positive maps according to how well they preserve entanglement with an environment. When dealing with bipartite systems it is important for practical applications, whether a certain quantum channel can be implemented by applying local quantum channels to the separate systems. This gives rise to the following definition:

Definition 8. (*Separable maps*)
A completely positive map $\mathcal{T} \in \mathfrak{CP}\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B}\right)$ is called ***separable*** *if it has a Kraus-decomposition of the form*

$$\mathcal{T}(X) = \sum_{i=1}^{r} (A_i \otimes B_i)^{\dagger} X (A_i \otimes B_i) ,$$

with $A_i \in \mathfrak{M}_{n_A,m_A}$ and $B_i \in \mathfrak{M}_{n_B,m_B}$. This means all Kraus-operators are product operators *with respect to the bipartition. We will use the notation:*

$$\mathfrak{Sep}\left(\mathfrak{M}_{n_A}\otimes\mathfrak{M}_{n_B},\mathfrak{M}_{m_A}\otimes\mathfrak{M}_{m_B}\right)=\left\{\mathcal{T}\in\mathfrak{CP}\left(\mathfrak{M}_{n_A}\otimes\mathfrak{M}_{n_B},\mathfrak{M}_{m_A}\otimes\mathfrak{M}_{m_B}\right):\mathcal{T}\ \textit{separable}\right\}$$

It would be nice, if the maps in $\mathfrak{Sep}\left(\mathfrak{M}_{n_A}\otimes\mathfrak{M}_{n_B},\mathfrak{M}_{m_A}\otimes\mathfrak{M}_{m_B}\right)$ could all be prepared in a physical setting by two parties, which communicate classically. Unfortunately this is not the case and we have to introduce the following set of maps separately.

Definition 9. (*Local operation and classical communication*)

*A quantum channel $\mathcal{T}\in\mathfrak{CPTP}\left(\mathfrak{M}_{n_A}\otimes\mathfrak{M}_{n_B},\mathfrak{M}_{m_A}\otimes\mathfrak{M}_{m_B}\right)$ is called an **k-round LOCC-map** with respect to the bipartition (A:B) if it has a Kraus-decomposition of the form*

$$\mathcal{T}(X)=\sum_{i_1^A,\dots,i_k^A,i_1^B,\dots,i_k^B=1}^{r_1^A,\dots,r_k^A,r_1^B,\dots,r_k^B}C\left(i_1^A,\dots,i_k^A,i_1^B,\dots,i_k^B\right)^\dagger XC\left(i_1^A,\dots,i_k^A,i_1^B,\dots,i_k^B\right)\ ,$$

with $C\left(i_1^A,\dots,i_k^A,i_1^B,\dots,i_k^B\right)\in\mathfrak{M}_{n_An_B,m_Am_B}$ of the form

$$C\left(i_1^A,\dots,i_k^A,i_1^B,\dots,i_k^B\right)=A_k^{i_k^Ai_{k-1}^B}A_{k-1}^{i_{k-1}^Ai_{k-2}^B}\cdots A_2^{i_2^Ai_1^B}A_1^{i_1^A}\otimes B_k^{i_k^Bi_k^A}B_{k-1}^{i_{k-1}^Bi_{k-1}^A}\cdots B_2^{i_2^Bi_2^A}B_1^{i_1^Ai_1^B}\ .$$

The $A_l^{ij}\in\mathfrak{M}_{n_A^l,m_A^l}$ and $B_l^{ij}\in\mathfrak{M}_{n_B^l,m_B^l}$ are such that, the completely positive maps defined by

$$X\mapsto\sum_i\left(A_l^{ij}\right)^\dagger XA_l^{ij}$$

and

$$X\mapsto\sum_i\left(B_l^{ij}\right)^\dagger XB_l^{ij}$$

are trace-preserving, i.e. they are quantum channels, for all $l\in\{1,\dots k\}$ and $j\in\{1,\dots k-1\}$. The dimensions fulfill

$$\begin{aligned}n_A^1&=n_A\ ,\\ m_A^k&=m_A\ ,\\ n_B^1&=n_B\ ,\\ m_B^k&=m_B\ ,\end{aligned}$$

and for all $l\in\{1,\dots k-1\}$

$$\begin{aligned}m_A^l&=n_A^{l+1}\ ,\\ m_B^l&=n_B^{l+1}\ .\end{aligned}$$

A quantum channel $\mathcal{T} \in \mathfrak{CPTP}(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B})$ is called an ***LOCC-map*** *with respect to the bipartition (A:B) if it is an k-round LOCC-map with respect to the bipartition (A:B) for some $k \in \mathbb{N}$.*

We will use the notation

$$\mathfrak{Locc}(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B}) = \{\mathcal{T} \in \mathfrak{CPTP}(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B}) : \mathcal{T} \text{ } LOCC\}$$

We can think about maps in $\mathfrak{Locc}(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B})$ in terms of separate channel-application and communication rounds. First Alice applies a quantum channel and a measurement to her system and sends information about the applied Kraus-operator and the measurement outcome to Bob. He can use this information to perform a quantum channel on his part of the system depending on what Alice did to her system. After doing so, he sends information about the applied Kraus-operator back to Alice and the whole process is repeated always using all the information from the previous round. If this is iterated for N rounds, we end up with an N-round LOCC operation.

Remark 5. *The set $\mathfrak{Locc}(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B})$ is far from well understood. It is however necessary for the theory in order to be applicable to the physical world. As shown in [CB99] [RD09] we have*

$$\overline{\mathfrak{Locc}(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B})} \subsetneq \mathfrak{Sep}(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B}) \ ,$$

for certain $n_A, n_B, m_A, m_B \in \mathbb{N}$, i.e. the LOCC-maps are a proper subset of the separable maps, and there are separable maps, that cannot be implemented by applying local maps and communicate classically, not even in the limit of infinite rounds. This gap becomes apparent already in very fundamental tasks, such as the discrimination of quantum states [RD09].

When we formulate our results, we will often use LOCC-maps in the definitions of certain quantities, but the proofs would work for arbitrary separable quantum channels. Note that if we would formulate the definitions using arbitrary separable quantum channels instead of LOCC-maps, they would not be meaningful in the physical theory anymore, which is why we use LOCC-maps in our definitions.

We have introduced the basic building blocks of quantum theory. In the next section we will introduce quantum capacities, which quantify how much information can be send through a quantum channel. These will lead to the mathematical questions that we want to study in the remaining parts of this thesis.

1.2. Channel Capacities

In the following we want to quantify how much quantum information can be send through a quantum channel. The quantities that we will define are called capacities and there are different capacities for the different physical settings.

To introduce different capacities in a concise manner, we define a general capacity, which can be seen as a prototype for the other capacities, which we will introduce. Defining a capacity requires certain formalizations to be made. First we have to formalize the information processing task in a mathematical precise way. Then we have to specify what resources we are allowed to use and how they enter in the mathematical framework. The definition of the general capacity states the mathematical framework and how certain resources enter in this picture. But it leaves the choice of the resources open, and different choices lead to different capacities.

Sending information from a physical system to another using a physical map, i.e. a quantum channel, can be stated as an approximation problem on the space of matrix maps. We use the given quantum channel and a proper encoding- and decoding scheme to approximate the identity channel between two reference-systems of the same dimension. The identity channel corresponds to a perfect map sending information from one reference-system to the other without loss. The dimension of the reference-systems fixes a choice of units in which we measure the information, that has been transferred. By using coding schemes that are general enough, we can choose the reference-systems on which the identity acts freely, but we will use the most simple case, i.e. a qubit system.

A major difference between quantum and classical theory is the existence of entangled states, which cannot be treated as separable states on many systems. It might be favorable for our approximation problem to apply many instances of the given quantum channel together to an entangled state living on many copies of the system. Afterwards we try to approximate some number of copies of the identity. To take this into account we define the capacity as a limit, where the number of copies used, goes to infinity.

Taken all this together, we get the following general definition of a capacity:

Definition 10. (*General Capacity*)

For a quantum channel $\mathcal{T} \in \mathfrak{CPTP}\left(\mathfrak{M}_n, \mathfrak{M}_m\right)$ a number $r \in \mathbb{R}^+$ is called an achievable rate, if there are sequences $\left(k^1_\nu\right)_\nu, \left(k^2_\nu\right)_\nu \in \mathbb{N}^\mathbb{N}$ with $k^1_\nu \to \infty$ for $\nu \to \infty$, such that $r = lim_{\nu\to\infty} \frac{k^2_\nu}{k^1_\nu}$ and we have

$$\Delta\left(k^1_\nu, k^2_\nu\right) := inf_{\mathcal{C}_\nu} \|\mathcal{C}_\nu\left[\mathcal{T}^{\otimes k^1_\nu}\right] - id_2^{\otimes k^2_\nu}\| \to 0$$

as $\nu \to \infty$. Here the infimum goes over all coding schemes

$$\mathcal{C}_\nu : \mathfrak{CPTP}\left(\mathfrak{M}_n^{\otimes k^1_\nu}, \mathfrak{M}_m^{\otimes k^1_\nu}\right) \to \mathfrak{CPTP}\left(\mathfrak{M}_2^{\otimes k^2_\nu}, \mathfrak{M}_2^{\otimes k^2_\nu}\right) ,$$

which are allowed by the specifications in the concrete case.

*We define the **capacity** of $\mathcal{T} \in \mathfrak{CPTP}\left(\mathfrak{M}_n, \mathfrak{M}_m\right)$ to be*

$$\mathcal{C}\left(\mathcal{T}\right) = sup\{r \in \mathbb{R}^+ : r \text{ is achievable rate}\} .$$

In the above definition we used an arbitrary norm $\|\bullet\|$ on the space of matrix maps. We will now come to two concrete cases, which will be important in the following.

Definition 11. (*Quantum Capacity $\mathcal{Q}$. [DK04]*)

For a quantum channel $\mathcal{T} \in \mathfrak{CPTP}(\mathfrak{M}_n, \mathfrak{M}_m)$ we define the quantum capacity $\mathcal{Q}(\mathcal{T})$ as the general capacity according to Definition 10, by using

$$\| \bullet \| = \| \bullet \|_{\diamond}$$

the $\diamond-$norm, see Appendix A for details, and by only allowing coding schemes

$$\mathcal{C} : \mathfrak{CPTP}\left(\mathfrak{M}_n^{\otimes k_\nu^1}, \mathfrak{M}_m^{\otimes k_\nu^1}\right) \rightarrow \mathfrak{CPTP}\left(\mathfrak{M}_2^{\otimes k_\nu^2}, \mathfrak{M}_2^{\otimes k_\nu^2}\right)$$

of the form:

$$\mathcal{C}\left[\mathcal{T}^{\otimes k_\nu^1}\right] = \mathcal{D} \circ \mathcal{T}^{\otimes k_\nu^1} \circ \mathcal{E}$$

for encoding and decoding quantum channels $\mathcal{E} \in \mathfrak{CPTP}\left(\mathfrak{M}_2^{\otimes k_\nu^2}, \mathfrak{M}_m^{\otimes k_\nu^1}\right)$ and $\mathcal{D} \in \mathfrak{CPTP}\left(\mathfrak{M}_m^{\otimes k_\nu^1}, \mathfrak{M}_2^{\otimes k_\nu^2}\right)$.

Note that the above definition is in Schrödinger picture. We can define the same quantity in Heisenberg picture, by replacing all occurring quantum channels by their adjoints, and use the cb-norm instead of the $\diamond-$norm. These norms are defined in Appendix A.

Remark 6. *As studied in [DK04] there are many equivalent ways of defining the quantum capacity $\mathcal{Q}$. We could have used another measure of distance, such as the fidelity, see [MN07], or put some constraints on the sequences $\left(k_\nu^1\right)_\nu, \left(k_\nu^1\right)_\nu$ which we have to test. An important choice is the dimension of the reference system on which the identity acts. In our case this is a qubit system. When choosing a d-dimensional reference system, we obtain a different capacity $\mathcal{Q}'$ which fulfills*

$$\mathcal{Q}'(\mathcal{T}) = \frac{1}{log(d)} \mathcal{Q}(\mathcal{T}) \ ,$$

as shown in [DK04].

The above definition gives the capacity for sending quantum information through a quantum channel, when only local coding schemes are allowed. Sometimes we are interested in a more general setting, where we allow coding schemes assisted by classical communication between the two parties. This will lead to the following more complicated 2-way assisted quantum capacity

Definition 12. (*Two-way assisted quantum capacity $\mathcal{Q}_2$*)

For a quantum channel $\mathcal{T} \in \mathfrak{CPTP}(\mathfrak{M}_n, \mathfrak{M}_m)$ we define a two-way assisted coding scheme

$$\mathcal{C}_{k_\nu^1, k_\nu^2} : \mathfrak{CPTP}(\mathfrak{M}_n, \mathfrak{M}_m) \rightarrow \mathfrak{CPTP}\left(\mathfrak{M}_2^{\otimes k_\nu^2}, \mathfrak{M}_2^{\otimes k_\nu^2}\right)$$

for $k_\nu^1, k_\nu^2 \in \mathbb{N}$, where k_ν^1 denotes the number of channel uses, by

$$\mathcal{C}_{k_\nu^1,k_\nu^2}\left[\mathcal{T}\right] = tr_W \circ \mathcal{L}_{AB}^{k_\nu^1+1} \circ \widehat{\mathcal{T}}_{k^1} \circ \mathcal{L}_{AB}^{k_\nu^1} \circ \cdots \circ \widehat{\mathcal{T}}_2 \circ \mathcal{L}_{AB}^2 \circ \widehat{\mathcal{T}}_1 \circ \mathcal{L}_{AB}^1 \circ \mathcal{E}_W \ ,$$

with

$$\widehat{\mathcal{T}}_l : \mathfrak{h}_W^A \otimes \mathfrak{h}_W^B \otimes \mathfrak{M}_2^{\otimes m_l^A} \otimes \mathfrak{M}_n^A \otimes \mathfrak{M}_2^{\otimes m_l^B} \to \mathfrak{h}_W^A \otimes \mathfrak{h}_W^B \otimes \mathfrak{M}_2^{\otimes m_l^A} \otimes \mathfrak{M}_m^B \otimes \mathfrak{M}_2^{\otimes m_l^B}$$
$$\widehat{\mathcal{T}}_l = id_W \otimes id_{m_l^A} \otimes \mathcal{T} \otimes id_{m_l^B} \ .$$

Here we identified the system A with the input side of the communication setting and the system B with the output side, i.e. the input space $\mathfrak{M}_n^A$ corresponds to A and the output space $\mathfrak{M}_m^B$ to B. We denote by $\mathfrak{h}_W^A$ and $\mathfrak{h}_W^B$ local working Hilbert spaces, which can be used by A and B to store classical and quantum information. The sequences $\left(m_l^A\right)_l$ and $\left(m_l^B\right)_l$ fulfill

$$m_{k_\nu^1}^A = m_1^B = 0$$
$$m_1^A = m_{k_\nu^1}^B = k_\nu^2 \ ,$$

and denote the number of qubit spaces at A and B.

Note that while there is only one application of $\mathcal{T}$ in every step of the protocol, the following maps can act on any number of local spaces, in order to provide for arbitrary coding schemes.

We have

$$\mathcal{L}_{AB}^l : \mathfrak{h}_W^A \otimes \mathfrak{h}_W^B \otimes \mathfrak{M}_2^{\otimes m_l^A} \otimes \mathfrak{M}_m^B \otimes \mathfrak{M}_2^{\otimes m_l^B} \to$$
$$\mathfrak{h}_W^A \otimes \mathfrak{h}_W^B \otimes \mathfrak{M}_2^{\otimes m_{l+1}^A} \otimes \mathfrak{M}_n^A \otimes \mathfrak{M}_2^{\otimes m_{l+1}^B} \ ,$$

which are LOCC-maps, with respect to the partition $(A : B)$, and provide for encoding and decoding operations as well as for classical communication between the two parties.

As a technicality we define the map

$$\mathcal{E}_W : \mathfrak{M}_2^{\otimes m_1^A} \to \mathfrak{h}_W^A \otimes \mathfrak{h}_W^B \otimes \mathfrak{M}_2^{\otimes m_1^A}$$
$$\mathcal{E}_W\left(X\right) = \frac{I_{W_A} \otimes I_{W_B} \otimes X}{dim\left(\mathfrak{h}_W^A\right) dim\left(\mathfrak{h}_W^B\right)} \ ,$$

in order to introduce the working Hilbert spaces initialized in the maximally mixed state. The partial trace tr_W traces these auxiliary spaces out in the end of the protocol.

In analogy to the general capacity, see Definition 10, we call $r \in \mathbb{R}^+$ an achievable rate, if there exist sequences $k_\nu^1, k_\nu^2 \to \infty$, such that $r = lim_{\nu\to\infty} \frac{k_\nu^2}{k_\nu^1}$ and

$$\Delta\left(k_\nu^1, k_\nu^2\right) = inf \| \mathcal{C}_{k_\nu^1, k_\nu^2}\left[\mathcal{T}\right] - id_2^{\otimes k_\nu^2} \|_\diamond \to 0\,,$$

*as $\nu \to \infty$, where the infimum is taken over all coding schemes $\mathcal{C}_{k_\nu^1, k_\nu^2}$ of the form defined above. We define the **two-way assisted quantum capacity** by*

$$\mathcal{Q}_2\left(\mathcal{T}\right) := sup\{r \in \mathbb{R}^+ \text{ achievable rate}\}\,.$$

As the above quantities are quite complicated it is useful to have efficiently computable upper bounds. Such bounds, if good enough, will provide tools for the study of quantum channels. Distilling pure entangled states from mixed states is another problem in quantum information theory. We will define this problem in the next section, and it will turn out to be strongly connected to the problem of finding capacity bounds.

1.3. Problem of Entanglement Distillation

Many protocols in quantum information, e.g. the teleportation protocol for the identity channel in Example 2, rely on a pure maximally entangled state shared between A and B. By the nature of quantum theory one often has to deal with statistical mixtures of pure states. The problem of entanglement distillation can be stated as follows:

Problem 1. *Given $k \in \mathbb{N}$ identical copies of a, possibly mixed, quantum state $X \in \mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}$ on a bipartite system. Is there an LOCC-protocol, which generates a pure maximally entangled state $\frac{\omega_2}{2}$ out of $X^{\otimes k}$?*

A state X for which there is such a protocol is called distillable. In order to be precise, we make the following definition

Definition 13. (*Distillability*)

A quantum state $X \in \mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}$ is called distillable, if for any $\epsilon \geq 0$ there is an $k \in \mathbb{N}$ and an $\mathcal{L} \in \mathfrak{Locc}\left(\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}\right)^{\otimes k}, \mathfrak{M}_2 \otimes \mathfrak{M}_2\right)$, such that

$$\|\frac{\omega_2}{2} - \mathcal{L}\left(X^{\otimes k}\right)\|_1 \leq \epsilon\,,$$

where $\frac{\omega_2}{2}$ is a maximally entangled pure state.

In order to quantify the distillability of a quantum state we make the following definition in analogy to a capacity, see Definition 10, for the task of approximating the maximally entangled state.

Definition 14. (*2-way assisted Distillability $\mathcal{D}_2$*)

*We call $r \in \mathbb{R}^+$ an **achievable rate** for the distillation of $X \in \mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}$ if there are sequences $\left(k^1_\nu\right)_{\nu\in\mathbb{N}}$, $\left(k^2_\nu\right)_{\nu\in\mathbb{N}}$ such that $\frac{k^1_\nu}{k^2_\nu} \to r$ and $k^2_\nu \to \infty$ for $\nu \to \infty$ and $\mathcal{L}_\nu \in \mathfrak{Locc}\left(\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}\right)^{\otimes k^2_\nu}, \left(\mathfrak{M}_2 \otimes \mathfrak{M}_2\right)^{\otimes k^1_\nu}\right)$, such that*

$$\| \left(\frac{\omega_2}{2}\right)^{\otimes k^1_\nu} - \mathcal{L}_\nu \left(X^{\otimes k^2_\nu}\right) \|_1 \to 0$$

for $\nu \to \infty$.

$$\mathcal{D}_2\left(X\right) = sup\{r \in \mathbb{R}^+ \text{ achievable rate}\} \, .$$

Note that this definition is compatible with the definition of distillability as $\mathcal{D}_2\left(X\right) > 0$ iff X is distillable.

The definitions of the quantities $\mathcal{D}_2$ and $\mathcal{Q}_2$, see Definition 12, are quite similar. Later we will show, that they share similar properties.

It is clear from the above definitions, that separable states cannot be distillable. Therefore one could expect that the distillable states coincide with the entangled ones. But the answer turns out to be not that easy.

The following quantity will be useful in the following:

Definition 15. (*Logarithmic negativity, See [GV02]*)

For a quantum state $X \in \mathcal{B}\left(\mathbb{C}^{m_A} \otimes \mathbb{C}^{m_B}\right)$ we call

$$\mathcal{LN}\left(X\right) = log\|X^{T_B}\|_1$$

*the **logarithmic negativity** of X.*

We will now prove that $\mathcal{LN}\left(X\right)$ provides an upper bound to the distillability $\mathcal{D}_2\left(X\right)$. Note that despite the complicate definition of $\mathcal{D}_2\left(X\right)$ the quantity $\mathcal{LN}\left(X\right)$ is quite easy to compute, what makes this upper bound very useful in practice.

Theorem 2. (*See [GV02]*)

For a quantum state $X \in \mathcal{B}\left(\mathbb{C}^{m_A} \otimes \mathbb{C}^{m_B}\right)$ we have

$$\mathcal{D}_2\left(X\right) \leq \mathcal{LN}\left(X\right) \, .$$

In particular we have that X is not distillable if $X^{T_B} \geq 0$

Proof. Take an achievable rate $r \in \mathbb{R}^+$ for the distillation of the quantum state $X \in \mathcal{B}\left(\mathbb{C}^{m_A} \otimes \mathbb{C}^{m_B}\right)$ and sequences $\left(k^1_\nu\right)_{\nu\in\mathbb{N}}$, $\left(k^2_\nu\right)_{\nu\in\mathbb{N}}$ such that $\frac{k^1_\nu}{k^2_\nu} \to r$ and $k^2_\nu \to \infty$ for $\nu \to \infty$. Denote by $\mathcal{L}_\nu \in \mathfrak{CPTP}\left(\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}\right)^{\otimes k^2_\nu}, \left(\mathfrak{M}_2 \otimes \mathfrak{M}_2\right)^{\otimes k^1_\nu}\right)$ distillation maps which are LOCC with respect to the partition (A:B), such that $\Delta\left(k^1_\nu, k^2_\nu\right) = \| \left(\frac{\omega_2}{2}\right)^{\otimes k^1_\nu} - \mathcal{L}_\nu\left(X^{\otimes k^2_\nu}\right) \|_1 \to 0$.

$$
\begin{aligned}
2^{k_\nu^1} &= \| \left(\frac{\mathbb{F}_2}{2}\right)^{\otimes k_\nu^1} \|_1 = \| \left(\left(\frac{\omega_2}{2}\right)^{\otimes k_\nu^1} - \mathcal{L}_\nu \left(X^{\otimes k_\nu^2} \right) \right)^{T_B} + \left(\mathcal{L}_\nu \left(X^{\otimes k_\nu^2} \right) \right)^{T_B} \|_1 \\
&\leq \| \left(\left(\frac{\omega_2}{2}\right)^{\otimes k_\nu^1} - \mathcal{L}_\nu \left(X^{\otimes k_\nu^2} \right) \right)^{T_B} \|_1 + \| \left(\mathcal{L}_\nu \left(X^{\otimes k_\nu^2} \right) \right)^{T_B} \|_1 \\
&\leq \| \left(\mathrm{id}_{m_A} \otimes \vartheta_{m_B}\right)^{\otimes k_\nu^1} \|_{1\to 1} \Delta \left(k_\nu^1, k_\nu^2\right) \\
&\qquad + \| \left[\left(\mathrm{id}_2 \otimes \vartheta_2\right)^{\otimes k_\nu^1} \circ \mathcal{L}_\nu \circ \left(\mathrm{id}_{m_A} \otimes \vartheta_{m_B}\right)^{\otimes k_\nu^2} \right] \left(\left(X^{T_B}\right)^{\otimes k_\nu^2} \right) \|_1 \\
&\leq \Delta \left(k_\nu^1, k_\nu^2\right) 2^{k_\nu^1} + \| \left(X^{T_B}\right)^{\otimes k_\nu^2} \|_1 \\
&= \Delta \left(k_\nu^1, k_\nu^2\right) 2^{k_\nu^1} + \| X^{T_B} \|_1^{k_\nu^2} ,
\end{aligned}
$$

where we used elementary properties of the transposition and of the trace-norm, such as submultiplicativity and tensor-submultiplicativity, see Appendix A for details. Furthermore we used that

$$
\left[\left(\mathrm{id}_2 \otimes \vartheta_2\right)^{\otimes k_\nu^1} \circ \mathcal{L}_\nu \circ \left(\mathrm{id}_{m_A} \otimes \vartheta_{m_B}\right)^{\otimes k_\nu^2} \right] \in \mathfrak{CPTP} \left(\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}\right)^{\otimes k_\nu^2}, \left(\mathfrak{M}_2 \otimes \mathfrak{M}_2\right)^{\otimes k_\nu^1} \right)
$$

and therefore fulfills $\| \left(\mathrm{id}_2 \otimes \vartheta_2\right)^{\otimes k_\nu^1} \circ \mathcal{L}_\nu \circ \left(\mathrm{id}_{m_A} \otimes \vartheta_{m_B}\right)^{\otimes k_\nu^2} \|_{1\to 1} = 1$.

From this we get

$$
\begin{aligned}
& 2^{k_\nu^1} \left(1 - \Delta \left(k_\nu^1, k_\nu^2\right)\right) \leq \| X^{T_B} \|_1^{k_\nu^2} \\
\Longrightarrow & \frac{k_\nu^1}{k_\nu^2} + \frac{\log \left(1 - \Delta \left(k_\nu^1, k_\nu^2\right)\right)}{k_\nu^2} \leq \log \left(\| X^{T_B} \|_1 \right)
\end{aligned}
$$

by taking the logarithm. In the limit $\nu \to \infty$ we get

$$
r \leq \log \left(\| X^{T_B} \|_1 \right) = \mathcal{LN} \left(X\right) ,
$$

which gives the desired result as this holds for all achievable rates $r \in \mathbb{R}^+$.

□

Remark 7. *We will call a state $X \in \mathcal{B}\left(\mathbb{C}^{m_A} \otimes \mathbb{C}^{m_B}\right)$ PPT if its partial transpose is positive, i.e. if $X^{T_B} \geq 0$. The above Theorem states, that PPT-states are not distillable. We call states NPT if they are not PPT.*

It can be shown [MH98] that there are entangled PPT-states, which means that there is entanglement in mixed states that cannot be used to distill pure entangled states. Such states are also called bound entangled as their entanglement cannot be used for applications requiring pure entanglement. Thus the characterization of distillable states is not that simple.

The next idea that one could have about the characterization of distillable states is that the converse of the above theorem could be true meaning that the distillable states are exactly the NPT states.

The question is, whether there are non-distillable NPT states, hence bound entangled. This is the famous NPT-bound-entanglement problem, which has been unknown since 1998 [MH99] and despite great effort and progress has not been solved yet.

In this thesis we want to look at this question from a different perspective. Instead of trying to find NPT states that are not distillable we try to find different criteria for a state to be not distillable. In particular we will search for maps other than the transposition, which generate such a criterion in the same way.

1.4. Problem of Quantum Capacity Bounds

The transposition provides a simple criterion for deciding whether a state is not distillable. There is another setting in quantum information theory where the transposition is useful. This is the following problem.

Problem 2. *Given a quantum channel $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$, find an efficiently computable upper bound on the quantum channel capacity $\mathcal{Q}(\mathcal{T})$ of the channel?*

It may be quite difficult to evaluate the quantum capacity $\mathcal{Q}(\mathcal{T})$, see Definition 10, for certain channels $\mathcal{T}$ directly. This is why an efficiently computable upper bound is useful for applications.

Surprisingly we get an easy upper bound, by using the transposition.

Theorem 3. (*See [DK04]*)

For a quantum channel $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ we have

$$\mathcal{Q}(\mathcal{T}) \leq log(\|\mathcal{T} \circ \vartheta_n\|_\diamond) \ ,$$

where ϑ denotes the transposition.

Proof. Take an achievable rate $r \in \mathbb{R}^+$ for the channel T and sequences $\left(k_\nu^1\right)_{\nu\in\mathbb{N}}$, $\left(k_\nu^2\right)_{\nu\in\mathbb{N}}$ such that $\frac{k_\nu^1}{k_\nu^2} \to r$ and $k_\nu^2 \to \infty$ for $\nu \to \infty$. Denote by $\mathcal{E}_\nu : \mathfrak{M}_2^{\otimes k_\nu^1} \to \mathfrak{M}_n^{\otimes k_\nu^2}$ and $\mathcal{D}_\nu : \mathfrak{M}_n^{\otimes k_\nu^2} \to \mathfrak{M}_2^{\otimes k_\nu^1}$ suitable encoding and decoding channels, such that $\Delta\left(k_\nu^1, k_\nu^2\right) = \|\mathrm{id}_2^{\otimes k_\nu^1} - \mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\|_\diamond \to 0$ for $\nu \to \infty$.

$$
\begin{aligned}
2^{k_\nu^1} = \|\vartheta_2^{\otimes k_\nu^1}\|_\diamond &= \| \left(\mathrm{id}_2^{\otimes k_\nu^1} - \mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\right) \circ \vartheta_2^{\otimes k_\nu^1} + \left(\mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\right) \circ \vartheta_2^{\otimes k_\nu^1} \|_\diamond \\
&\leq \| \left(\mathrm{id}_2^{\otimes k_\nu^1} - \mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\right) \circ \vartheta_2^{\otimes k_\nu^1} \|_\diamond + \| \left(\mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\right) \circ \vartheta_2^{\otimes k_\nu^1} \|_\diamond \\
&\leq \Delta\left(k_\nu^1, k_\nu^2\right) 2^{k_\nu^1} + \|\mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \vartheta_n^{\otimes k_\nu^2} \circ \vartheta_n^{\otimes k_\nu^2} \circ \mathcal{E}_\nu \circ \vartheta_2^{\otimes k_\nu^1} \|_\diamond \\
&\leq \Delta\left(k_\nu^1, k_\nu^2\right) 2^{k_\nu^1} + \| \left(\mathcal{T} \circ \vartheta_n\right)^{\otimes k_\nu^2} \|_\diamond \\
&= \Delta\left(k_\nu^1, k_\nu^2\right) 2^{k_\nu^1} + \|\mathcal{T} \circ \vartheta_n\|_\diamond^{k_\nu^2} ,
\end{aligned}
$$

where we used submultiplicativity of the cb-norm, the fact that $\|\mathcal{D}\|_\diamond = 1$ for a quantum channel $\mathcal{D}$, that $\vartheta_n \circ \mathcal{E} \circ \vartheta_2$ is again a quantum channel if $\mathcal{E}$ is a quantum channel and that the cb-norm fulfills $\|\mathcal{T}_1 \otimes \mathcal{T}_2\|_\diamond = \|\mathcal{T}_1\|_\diamond \|T_2\|_\diamond$ for all T_1, T_2.

From this we get

$$
\begin{aligned}
& 2^{k_\nu^1}\left(1 - \Delta\left(k_\nu^1, k_\nu^2\right)\right) \leq \|\mathcal{T} \circ \vartheta_n\|_\diamond^{k_\nu^2} \\
\implies & \frac{k_\nu^1}{k_\nu^2} + \frac{\log\left(1 - \Delta\left(k_\nu^1, k_\nu^2\right)\right)}{k_\nu^2} \leq \log\left(\|\mathcal{T} \circ \vartheta_n\|_\diamond\right)
\end{aligned}
$$

by taking the logarithm. In the limit $\nu \to \infty$ we get

$$
r \leq \log\left(\|\mathcal{T} \circ \vartheta_n\|_\diamond\right) ,
$$

which gives the desired result as this holds for all achievable rates $r \in \mathbb{R}^+$.

□

In the following we try to find other maps, which give a similar bound. As we will see, this question is connected to our earlier question on criteria for distillability and we will clarify this relation in the discussion.

To conclude our introduction we give an outline of the remaining part of this work. In Chapter 2 we will introduce some tools from matrix theory and convex analysis, namely linear preserver problems and mapping cones, which we will use to characterize certain matrix maps arising in quantum information theoretical applications. In Chapter 3 we will present our results concerning the problem of entanglement distillation and in the Chapter 4 we will discuss our results for the construction of new bounds on the quantum capacity of a quantum channel. In Chapter 5 we close with a conclusion.

2. Preliminaries

In the previous chapter we gave an introduction to the formalism of quantum information theory. We also stated our main goals namely to find upper bounds on the distillability and the channel capacity similar to the ones coming from the transposition. One strategy to tackle this problems is, to identify properties of the transposition, which have been used to prove the known bounds. Starting from such properties we try to construct different maps, for which the proofs also work. We will need some further theory, which can be used to characterize such maps.

In the first part of this chapter we will introduce linear preserver problems, which provide many useful characterization results. We will use the second part of this chapter to give an introduction to the theory of mapping cones. This theory gives a formalism to classify the different classes of positive maps introduced in chapter 1 and to show the connections between those classes.

2.1. Linear Preservers and Wigner's Theorem

The theory of linear preserver problems (LPP) provides many useful characterization theorems and the transposition plays a special role in most of them. We will try to understand the role of the transposition in this context. From this it will become clear, why the transposition appears in such a variety of applications. For further information on linear preserver problems see [SP92].

2.1.1. Automorphisms on $\mathbb{C}$

Before introducing LPPs we need some elementary results about the automorphisms on $\mathbb{C}$, which we will prove in this section. In particular we will need, that the only continuous automorphisms on $\mathbb{C}$ are the identity and complex conjugation. This can be seen as the reason, why the transposition appears in the LPPs that we are going to study. We will follow some argument in [Yal66].

Lemma 2. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a continuous automorphism, then $\phi|_{\mathbb{R}} = id_{\mathbb{R}}$.*

Proof. It is clear that every subfield of $\mathbb{C}$ contains $\mathbb{Q}$ as a subfield. Now take the set $F = \{a \in \mathbb{C} : \phi(a) = a\}$ of fix points of ϕ. An easy calculation shows, that F is a subfield of $\mathbb{C}$ and thus $\mathbb{Q} \subset F$. By continuity of ϕ, F has to be closed, and thus by taking the closure of $\mathbb{Q} \subset F$, we obtain $\mathbb{R} \subset F$. □

With this lemma we are able to prove the following characterization of continuous automorphisms on $\mathbb{C}$.

Theorem 4. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a continuous automorphism. Then $\phi(\mathbb{R}) \subset \mathbb{R}$ and we have either $\phi = id_{\mathbb{C}}$ or $\phi = \overline{id_{\mathbb{C}}}$.*

Proof. By Lemma 2 $\mathbb{R}$ is left pointwise invariant by ϕ. Applying ϕ to $i^2 = -1$ gives $\phi(i)^2 = -1$ and therefore we have either $\phi(i) = i$ or $\phi(i) = -i$. An application of ϕ to $a + ib \in \mathbb{C}$ for $a, b \in \mathbb{R}$, together with the later results of the pointwise invariance of $\mathbb{R}$ under ϕ and $\phi(i) = i$ or $\phi(i) = -i$ finishes the proof. □

The above theorem can be generalized further and one can show, that the only isomorphisms between subfields of $\mathbb{C}$ for which $\mathbb{R}$ is contained in their domains and mapped to itself are $\mathrm{id}_{\mathbb{R}}$, $\mathrm{id}_{\mathbb{C}}$ or complex conjugation [Yal66]. In this generalization the requirement of the isomorphism to be continuous has been dropped.

This generalization together with the above theorem shows that a discontinuous automorphisms cannot map $\mathbb{R}$ to itself. Of course it has to map a dense subset of $\mathbb{R}$, namely $\mathbb{Q}$, to itself. It can be shown that such an automorphism maps $\mathbb{R}$ to a dense subset of the whole complex plane. This behavior justifies, why the discontinuous automorphisms of $\mathbb{C}$ are also called "wild automorphisms" [Yal66].

2.1.2. Linear Preserver Problems

The characterization of automorphisms on $\mathbb{C}$ in the last subsection showed, that there are just two continuous ones, the identity and complex conjugation. In this chapter we will relate this to certain problems stated on matrix spaces.

Remember that we want to characterize maps, that can be used in a similar way as the transposition to obtain upper bounds on distillability or the quantum capacities. Therefore we are interested in maps on certain matrix spaces over $\mathbb{C}$ and in particular on maps on hermitian matrices as quantum states are necessarily hermitian. For hermitian matrices complex conjugation is the same as transposition, which shows, that the result about automorphisms on $\mathbb{C}$ could be of interest here.

We will introduce a certain class of characterization problems, called linear preserver problems. A linear preserver problem [SP92] asks about the characterization of all linear maps between certain matrix spaces, which preserve certain properties of the matrices, such as rank, determinant, etc..

The choice of the preserved property can lead to a surprising reduction of possible maps. In the cases that we consider here, there will be just two possibilities connecting to the identity and the transposition up to certain transformations. The results about automorphisms on $\mathbb{C}$ can be seen as the reason for this behavior and the connection can be established by the application of the fundamental theorem of projective geometry, see Appendix B.

The results proved here can be used later to show, that certain properties are too strong, in the sense that they reduce the possibilities of maps fulfilling them to a small and trivial set.

The following linear preserver problem will be the foundation of our study and we build most of the theory on it. In the argumentation we will follow [JH85].

Theorem 5. (*Rank-1 preserver. [JH85]*)
Assume that $\mathcal{T} \in \mathfrak{H}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_n)$ is a bijective map that preserves rank 1, i.e. we have

$$\mathit{rank}(\mathcal{T}(X)) = 1$$

for all $X \in \mathfrak{M}_n$ with $\mathit{rank}(X) = 1$

Then there is an invertible S such that either $\mathcal{T}(X) = \epsilon S X S^\dagger$ or $\mathcal{T}(X) = \epsilon S X^T S^\dagger$ for $\epsilon = \pm 1$.

Proof. We will prove this result for the restricted map $\mathcal{T} : \mathfrak{H}_n \to \mathfrak{H}_n$ first. The proof will rely on an identification of rank-1 idempotents $|u\rangle\langle u| \in \mathfrak{M}_n$ with elements in the projective space $[u] \in \mathfrak{P}_n$, see Appendix B. With this identification in mind we will define the following map between projective spaces:

$$\widehat{\mathcal{T}} : \mathfrak{P}_0(\mathbb{C}^n) \to \mathfrak{P}_0(\mathbb{C}^n) \tag{2.1}$$

$$\widehat{\mathcal{T}}([u]) = [v] \ , \tag{2.2}$$

with u and v such that $\mathcal{T}(|u\rangle\langle u|) = \epsilon |v\rangle\langle v|$ for a sign $\epsilon = \pm 1$, which we will show to be unique. We identify 0 with the nopoint in projective space and as $\mathcal{T}$ is linear, which implies $\mathcal{T}(0) = 0$, we can set $\widehat{\mathcal{T}}([0]) = [0]$.

The remaining part of the proof shows that the map $\widehat{\mathcal{T}}$ is a morphism on projective spaces and this will allow us to apply the fundamental theorem of projective geometry to finish our characterization.

As $\mathcal{T}$ is hermiticity preserving there exists a unique sign $\epsilon = \pm 1$ such that for all $u \in \mathbb{C}^n$ we find a $v \in \mathbb{C}^n$ with $\mathcal{T}(|u\rangle\langle u|) = \epsilon |v\rangle\langle v|$. The uniqueness of $\epsilon = \pm 1$ can be seen by taking the function

$$\sigma : \mathfrak{S}^n \to \mathbb{R}$$
$$\sigma(|u\rangle) = \lambda(\mathcal{T}(|u\rangle\langle u|))$$

acting on the unit sphere $\mathfrak{S}^n \subset \mathbb{C}^n$ and with $\lambda(X) = \lambda_{\max}(X) + \lambda_{\min}(X)$ the sum of the minimal and maximal eigenvalue of $X \in \mathfrak{M}_n$. A sign change would correspond to the existence of $|u_1\rangle, |u_2\rangle \in \mathfrak{S}^n$ such that $\sigma(|u_1\rangle) > 0$ and $\sigma(|u_2\rangle) < 0$. As $\lambda_{\max}$ and $\lambda_{\min}$ are continuous functions, σ is a continuous function as well. By an application of the mean value theorem to a path connecting $|u_1\rangle$ and $|u_2\rangle$ there is a $|u\rangle \in \mathfrak{S}^n$ for which $f(|u\rangle) = 0$ and thus rank $(\mathcal{T}(|u\rangle\langle u|)) \neq 1$ which contradicts the rank-1 preserver property of $\mathcal{T}$.

In order to apply the fundamental theorem we have to show that $\widehat{\mathcal{T}}$ is a collineation , see Definition 29.

As $\mathcal{T}$ is linear we have $\mathcal{T}(0) = 0$ and thus by identifying 0 with the nopoint in projective space the first property of Definition 29 follows. To show the second property, we have to show that

$$\widehat{\mathcal{T}}([u_1] \vee [u_2]) = \widehat{\mathcal{T}}([u_1]) \vee \widehat{\mathcal{T}}([u_2])$$

for all $[u_1], [u_2] \in \mathfrak{P}_0(\mathbb{C})$. If either $[u_1] = [0]$ or $[u_2] = [0]$ this is clear by the definition of $\widehat{\mathcal{T}}$.

In the remaining cases take $[u_3] \in \mathfrak{P}(\mathbb{C}) \setminus \{0\}$ with $[u_3] \in [u_1] \vee [u_2]$. We have to show that $\widehat{\mathcal{T}}([u_3]) \in \widehat{\mathcal{T}}([u_1]) \vee \widehat{\mathcal{T}}([u_2])$. By the definition of projective space, we have $|u_1\rangle, |u_2\rangle, |u_3\rangle \in \mathbb{C}^n$ corresponding to the $[u_k]$. And by assumption there are $c_1, c_2 \in \mathbb{C}$ such that $|u_3\rangle = c_1 |u_1\rangle + c_2 |u_2\rangle$. With $c := c_1\overline{c_2}$ we have:

$$\begin{aligned}
\epsilon |v_3\rangle \langle v_3| &= \mathcal{T}(|u_3\rangle \langle u_3|) && (2.3)\\
&= \mathcal{T}((c_1 |u_1\rangle + c_2 |u_2\rangle)(\overline{c_1} \langle u_1| + \overline{c_2} \langle u_2|)) && (2.4)\\
&= |c_1|^2 \mathcal{T}(|u_1\rangle \langle u_1|) + \mathcal{T}(c |u_1\rangle \langle u_2| + \overline{c} |u_2\rangle \langle u_1|) + |c_2|^2 \mathcal{T}(|u_2\rangle \langle u_2|) && (2.5)\\
&= |c_1|^2 \epsilon |v_1\rangle \langle v_1| + \mathcal{T}(c |u_1\rangle \langle u_2| + \overline{c} |u_2\rangle \langle u_1|) + |c_2|^2 \epsilon |v_2\rangle \langle v_2| \ . && (2.6)
\end{aligned}$$

Clearly we have

$$\operatorname{rank}(c |u_1\rangle \langle u_2| + \overline{c} |u_2\rangle \langle u_1|) \leq 2 \tag{2.7}$$

and as this matrix is hermitian there are projectors $|w_1\rangle \langle w_1|, |w_2\rangle \langle w_2|$ such that

$$c |u_1\rangle \langle u_2| + \overline{c} |u_2\rangle \langle u_1| = r_1 |w_1\rangle \langle w_1| + r_2 |w_2\rangle \langle w_2| \ . \tag{2.8}$$

with $r_1, r_2 \in \mathbb{R}$. Together with the above calculation this implies that

$$\begin{aligned}
\operatorname{rank}\left(\epsilon |v_3\rangle \langle v_3| - |c_1|^2 \epsilon |v_1\rangle \langle v_1| - |c_2|^2 \epsilon |v_2\rangle \langle v_2|\right) &= \operatorname{rank}(\mathcal{T}(c |u_1\rangle \langle u_2| + \overline{c} |u_2\rangle \langle u_1|)) && (2.9)\\
&= \operatorname{rank}(\mathcal{T}(r_1 |w_1\rangle \langle w_1|) + \mathcal{T}(r_2 |w_2\rangle \langle w_2|)) && (2.10)\\
&\leq 2 \, , && (2.11)
\end{aligned}$$

where we have used, that $\mathcal{T}$ is a linear rank-1 preserver. But this means that v_1, v_2, v_3 have to be linearly dependent. Formulated in projective space this means that $\widehat{\mathcal{T}}([u_3]) \in \widehat{\mathcal{T}}([u_1]) \vee \widehat{\mathcal{T}}([u_2])$ and we showed $\widehat{\mathcal{T}}$ to be a collinear map between projective spaces. As it is bijective it is also a collineation.

Now we can use the fundamental theorem of projective geometry to get the following representation:

$$\widehat{\mathcal{T}}([u]) = [f(|u\rangle)] \ , \tag{2.12}$$

where $f : \mathbb{C}^n \to \mathbb{C}^n$ is a semilinear map, see Appendix B. As $\widehat{\mathcal{T}}$ is linear and bijective, we know that f has to be linear and bijective. The homomorphism on the field $\mathbb{C}$ used to define the semilinear map f has to be linear and bijective too. Thus it is a continuous automorphism and according to Theorem 4 the only possibilities for this automorphism are the identity and the complex conjugation. Collecting everything together there exists an invertible matrix S, such that we have either $\widehat{\mathcal{T}}([u]) = [S|u\rangle]$ or $\widehat{\mathcal{T}}([u]) = \left[S\overline{|u\rangle}\right]$. To get the corresponding representation of T, we just use the spectral decomposition for arbitrary $A \in \mathfrak{H}_n$ to get either

$$\begin{aligned}
\mathcal{T}(A) &= \sum_i r_i \mathcal{T}(|u_i\rangle\langle u_i|) && (2.13)\\
&= \epsilon \sum_i r_i S |u_i\rangle \langle u_i| S^\dagger && (2.14)\\
&= \epsilon S A S^\dagger && (2.15)\\
&\text{or} && (2.16)\\
&= \epsilon \sum_i r_i S \overline{|u_i\rangle\langle u_i|} S^\dagger && (2.17)\\
&= \epsilon S \overline{A} S^\dagger = \epsilon S A^T S^\dagger \ . && (2.18)
\end{aligned}$$

Therefore we proved the result for hermitian matrices.

To get the result on the whole $\mathfrak{M}_n$ just note that each matrix $A \in \mathfrak{M}_n$ can by Cartesian decomposition be written as

$$A = \frac{A + A^\dagger}{2} + i\frac{A - A^\dagger}{2i} \ ,$$

where $\frac{A+A^\dagger}{2}$ and $\frac{A-A^\dagger}{2i}$ are hermitian. This immediately gives the result.

□

The characterization of linear rank-1 preserver is quite useful and we will now prove some corollaries.

Corollary 5. (*See [Wol12]*)

Let $\mathcal{T} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_n)$ be a bijective map, with $\mathcal{T}^{-1} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_n)$. Then there is an invertible $S \in \mathfrak{M}_n$ such that either $\mathcal{T}(X) = SXS^\dagger$ or $\mathcal{T}(X) = SX^TS^\dagger$.

Proof. We will show first that $\mathcal{T}|_{\mathfrak{H}_n}$ is a rank-1 preserver and then apply the techniques from the proof of Theorem 5. Note first that as $\mathcal{T}$ is positive it is also hermiticity preserving and therefore $\mathcal{T}|_{\mathfrak{H}_n} : \mathfrak{H}_n \to \mathfrak{H}_n$ is well defined.

Assume that $\mathcal{T}|_{\mathfrak{H}_n}$ is not a rank-1 preserver. Then there is a $|u\rangle \in \mathbb{C}^n$ with

$$\mathcal{T}\left(|u\rangle\langle u|\right) = \sum_{i=1}^{r} k_i \,|u_i\rangle\langle u_i| \tag{2.19}$$

according to the spectral decomposition with rank $r > 1$ and $k_i \in \mathbb{R}^+$, because of positivity. But then applying $\mathcal{T}^{-1}$ gives the decomposition

$$\sum_{i=1}^{r} k_i \mathcal{T}^{-1}\left(|u_i\rangle\langle u_i|\right) = |u\rangle\langle u| \ , \tag{2.20}$$

with $\mathcal{T}^{-1}\left(|u_i\rangle\langle u_i|\right) \geq 0$. This gives the desired contradiction if $r > 1$ as $|u\rangle\langle u|$ cannot be decomposed in a sum of more than one positive term which are not all equal.

We have shown that $\mathcal{T}|_{\mathfrak{H}_n} : \mathfrak{H}_n \to \mathfrak{H}_n$ is a rank-1 preserver and therefore we are able to do the same reasoning as in the proof of Theorem 5. There is an invertible S such that either $\mathcal{T}|_{\mathfrak{H}_n}(X) = SXS^\dagger$ or $\mathcal{T}|_{\mathfrak{H}_n}(X) = SX^TS^\dagger$ for all $X \in \mathfrak{H}_n$. Note that the sign is fixed by assuming $\mathcal{T}$ to be positive.

The result follows immediately by decomposing an arbitrary matrix $A \in \mathfrak{M}_n$ in a linear combination of hermitian matrices as in the proof of Theorem 5.

□

Note that in the special case where $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_n$ is also trace-preserving, the above simplifies to unitary $S \in \mathfrak{U}_n$. In the following we will also need the famous theorem by Wigner:

Corollary 6. (*Wigner's Theorem. [Mol98]*)
Let $\mathcal{T} : \mathfrak{H}_n \to \mathfrak{H}_n$ be a map (not necessarily linear) that has the property

$$tr(\mathcal{T}(X)\,\mathcal{T}(Y)) = tr(XY) \tag{2.21}$$

for all matrices $X, Y \in \mathfrak{H}_n$ and maps rank-1 hermitian matrices bijective to rank-1 hermitian matrices. Then there is an unitary matrix $U \in \mathfrak{U}_n$ such that we have either $\mathcal{T}(X) = UXU^\dagger$ or $\mathcal{T}(X) = UX^TU^\dagger$.

Proof. Note that T has to be linear by the Wigner property, as we have

$$\operatorname{tr}\left(\mathcal{T}\left(X\right)\mathcal{T}\left(Y\right)\right) = \operatorname{tr}\left(XY\right) \ . \tag{2.22}$$

$$\begin{aligned}\operatorname{tr}\left(\mathcal{T}\left(X+Y\right)P\right) &= \operatorname{tr}\left(\left(X+Y\right)\mathcal{T}^{-1}\left(P\right)\right)\\ &= \operatorname{tr}\left(X\mathcal{T}^{-1}\left(P\right)\right) + \operatorname{tr}\left(Y\mathcal{T}^{-1}\left(P\right)\right)\\ &= \operatorname{tr}\left(\left(\mathcal{T}\left(X\right)+\mathcal{T}\left(Y\right)\right)P\right)\end{aligned}$$

for all rank-1 projectors P and as the set of hermitian matrices is spanned by the rank-1 projectors we can conclude that $\mathcal{T}$ is linear.

Then by Theorem 5 about the rank-1 preservers, we get the existence of an invertible matrix $S \in \mathfrak{M}_n$, such that either $\mathcal{T}\left(X\right) = SXS^\dagger$ or $\mathcal{T}\left(X\right) = SX^TS^\dagger$. By the Wigner property we obtain

$$\operatorname{tr}\left(S^\dagger SS^\dagger SY\right) = \operatorname{tr}\left(SI_nS^\dagger SYS^\dagger\right) = \operatorname{tr}\left(I_nY\right) = \operatorname{tr}\left(Y\right)$$

for all $Y \in \mathfrak{H}_n$. This is only possible if $S^\dagger SS^\dagger S = I_n$, which by positivity of $S^\dagger S$ leads to $S^\dagger S = I_n$. The case of $\mathcal{T}\left(X\right) = SX^TS^\dagger$ can be treated in analogy. This finishes the proof.

□

Our next goal is to characterize the maps of the following form:

Definition 16. (*Jordan Automorphism. [Sem08]*)

A linear bijective map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_n$ is called ***Jordan Automorphism*** *if it satisfies $\mathcal{T}\left(X^2\right) = \mathcal{T}\left(X\right)^2$.*

We will start with a lemma about basic properties of Jordan automorphisms, where we will follow the argumentation in [Her56].

Lemma 3. (*See [Her56]*)

Every Jordan automorphism $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_n$ has the following properties:

1. $\mathcal{T}\left(XB+BX\right) = \mathcal{T}\left(X\right)\mathcal{T}\left(B\right) + \mathcal{T}\left(B\right)\mathcal{T}\left(X\right)$.
2. $\mathcal{T}\left(XBX\right) = \mathcal{T}\left(X\right)\mathcal{T}\left(B\right)\mathcal{T}\left(X\right)$.

for all $X, B \in \mathfrak{M}_n$.

Proof. By doing the calculations we get:

1: Just evaluate the expression: $\mathcal{T}\left(\left(X+B\right)^2\right) = \mathcal{T}\left(\left(X+B\right)\right)^2$.

2: By 1. we have

$$
\begin{aligned}
\mathcal{T}\left(\left(XB+BX\right)X+X\left(XB+BX\right)\right) &= \mathcal{T}\left(XB+BX\right)\mathcal{T}\left(X\right)+\mathcal{T}\left(X\right)\mathcal{T}\left(XB+BX\right)\\
&= \left[\mathcal{T}\left(X\right)\mathcal{T}\left(B\right)+\mathcal{T}\left(B\right)\mathcal{T}\left(X\right)\right]\mathcal{T}\left(X\right)\\
&+\mathcal{T}\left(X\right)\left[\mathcal{T}\left(X\right)\mathcal{T}\left(B\right)+\mathcal{T}\left(B\right)\mathcal{T}\left(X\right)\right]\ .
\end{aligned}
$$

We also have

$$
\begin{aligned}
\mathcal{T}\left(\left(XB+BX\right)X+X\left(XB+BX\right)\right) &= \mathcal{T}\left(X^2B+2XBX+BX^2\right)\\
&= \mathcal{T}\left(B\right)\mathcal{T}\left(X\right)^2+\mathcal{T}\left(X\right)^2\mathcal{T}\left(B\right)+2\mathcal{T}(XBX)\ ,
\end{aligned}
$$

where we also used 1. and the defining property of Jordan automorphisms. From these equations we immediately obtain $\mathcal{T}\left(XBX\right)=\mathcal{T}\left(X\right)\mathcal{T}\left(B\right)\mathcal{T}\left(X\right)$

□

We are now able to prove the following characterization theorem for Jordan automorphisms.

Theorem 6. (*See [Sem08]*)
Let $\mathcal{T}:\mathfrak{M}_n\to\mathfrak{M}_n$ be a Jordan automorphism then there exists an invertible matrix $S\in M_n$ such that either $\mathcal{T}\left(X\right)=SXS^{-1}$ or $\mathcal{T}\left(X\right)=SX^TS^{-1}$.

Proof. We will show, that every Jordan automorphisms is also a hermiticity preserving rank-1 preserver. Then the result follows by applying Theorem 5.

Note that $\operatorname{rank}\left(X\right)=1$ iff $XBX\in\operatorname{span}\left(X\right)$ for all $B\in\mathfrak{M}_n$. By Lemma 3 we get

$$
\mathcal{T}\left(X\right)\mathcal{T}\left(Y\right)\mathcal{T}\left(X\right)=\mathcal{T}\left(XBX\right)\in\operatorname{span}\left(\mathcal{T}\left(X\right)\right)\ ,
$$

whenever $\operatorname{rank}\left(X\right)=1$. By bijectivity of $\mathcal{T}$, we have $\operatorname{rank}\left(\mathcal{T}\left(X\right)\right)=1$.

We also obtain

$$
\mathcal{T}\left(P\right)=\mathcal{T}\left(P^2\right)=\mathcal{T}\left(P\right)^2
$$

and by our reasoning we know, that rank-1 projectors are mapped to rank-1 projectors. Together with linearity this shows, that $\mathcal{T}$ is hermiticity preserving. The proof is finished by applying Theorem 5.

□

Remark 8. *From this theorem we see, that every Jordan automorphism is either an automorphism in the standard sense, meaning that $\mathcal{T}\left(XY\right)=\mathcal{T}\left(X\right)\mathcal{T}\left(Y\right)$, or an anti-automorphism, which means that we have $\mathcal{T}\left(XY\right)=\mathcal{T}\left(Y\right)\mathcal{T}\left(X\right)$.*

We are now in the position to characterize all linear maps that preserve unitary equivalence. We say that $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_n$ preserves unitary equivalence, if $\mathcal{T}(X) \sim_U \mathcal{T}(Y)$ is fulfilled whenever $X \sim_U Y$ for $X, Y \in \mathfrak{M}_n$. Here $X \sim_U Y$ means, that there exists a unitary matrix U, such that $UXU^\dagger = Y$.

When we introduced the implementation of a quantum channel via teleportation in Example 2, we saw the importance of unitary equivalence preserving quantum channels. These can be implemented via teleportation with probability $p_s = 1$.

In order to characterize these maps some differential geometry is needed, which is provided in Appendix C.

We will follow the argumentation in [Hia87] and start with a technical lemma.

Lemma 4. *(See [Hia87])*

Let $\mathcal{T} : \mathfrak{H}_n \to \mathfrak{H}_n$ be a unitary equivalence preserving linear map. Then either $\ker(\mathcal{T}) \subseteq \mathbb{R}I$ or $\ker(\mathcal{T}) \supseteq \ker(tr) \cap \mathfrak{H}_n$ holds.

Proof. Assume $\ker(\mathcal{T}) \setminus \mathbb{R}I \neq \emptyset$. We have to show that $\ker(\mathcal{T}) \supseteq \ker(\mathrm{tr}) \cap \mathfrak{H}_n$. Take a matrix $A \in \ker(\mathcal{T}) \setminus \mathbb{R}I$. As $\ker(\mathcal{T})$ is invariant under unitary transformations, we can without loss of generality assume A to be diagonal using the spectral theorem. Therefore we consider

$$A = \mathrm{diag}(r_1, \ldots, r_n) \ ,$$

with $r_k \in \mathbb{R}$ for all k and $r_1 \neq r_2$.

Note that with respect to the computational basis of $\mathbb{C}^n$ and after some unitary basis transformation we have,

$$\begin{aligned} &|1\rangle\langle 1| - |i\rangle\langle i| = \\ &\frac{1}{r_1 - r_2}\left(\mathrm{diag}(r_1, r_i, r_3, \ldots, r_{i-1}, r_2, r_{i+1}, \ldots, r_n) - \mathrm{diag}(r_2, r_i, r_3, \ldots, r_{i-1}, r_1, r_{i+1}, \ldots, r_n)\right) \end{aligned}$$

and therefore by linearity $|1\rangle\langle 1| - |i\rangle\langle i| \in \ker(\mathcal{T})\ \forall i \in \{1, \ldots, n\}$. For $X \in \ker(\mathrm{tr}) \cap \mathfrak{H}_n$, we can assume, after some unitary transformation, $X = \mathrm{diag}(x_1, \ldots, x_n)$ with $x_1 = -\sum_{i>1} x_i$ as $\mathrm{tr}(X) = 0$. From this we obtain the decomposition

$$X = -\sum_{i=2}^{n} x_i \left(|1\rangle\langle 1| - |i\rangle\langle i|\right) \in \ker(\mathcal{T}) \ .$$

Using the above decomposition leads to $\ker(\mathrm{tr}) \cap \mathfrak{H}_n \subseteq \ker(\mathcal{T})$ and we are done.

□

Note that in the case $\ker(\mathcal{T}) \subseteq \mathbb{R}I$ we have, by linearity, either $\ker(\mathcal{T}) = \mathbb{R}I$ or $\ker(\mathcal{T}) = \{0\}$. The latter one would imply that T is bijective. Now we are in the position to prove the characterization for unitarity equivalence preserving maps.

Theorem 7. (*See [Hia87]*)

Let $\mathcal{T}:\mathfrak{H}_n\to\mathfrak{H}_n$ be a linear map, which preserves unitary equivalence, then either there is an $A\in\mathfrak{H}_n$ such that

$$\mathcal{T}(X)=tr(X)\,A,\ \forall X\in\mathfrak{H}_n$$

or there are $a,b\in\mathbb{R}$ and a unitary matrix U, such that

$$\begin{aligned}\mathcal{T}(X)&=a\,tr(X)\,I+bUXU^\dagger,\ \forall X\in\mathfrak{H}_n\\ &or\\ \mathcal{T}(X)&=a\,tr(X)\,I+bUX^TU^\dagger,\ \forall X\in\mathfrak{H}_n\,.\end{aligned}$$

Proof. First we will show that $\mathcal{T}$ can be assumed bijective and unital without loss of generality.

By Lemma 4 we have two cases. In the first case $\ker(\mathcal{T})\supseteq\ker(\mathrm{tr})\cap\mathfrak{H}_n$ we have $\mathcal{T}(X)=\mathrm{tr}(X)\,A$ for some hermitian A.

For the second case $\ker(\mathcal{T})\subseteq\mathbb{R}I$ we see, that there are, by linearity, the possibilities $\ker(\mathcal{T})=\mathbb{R}I$ or $\ker(\mathcal{T})=\{0\}$. In the latter case T is bijective. If this is not the case, we have $\ker(\mathcal{T})=\mathbb{R}I$ and we can consider the map $\widehat{\mathcal{T}}:\mathfrak{H}_n\to\mathfrak{H}_n$ given by $\widehat{\mathcal{T}}(X)=\mathrm{tr}(X)\,I+\mathcal{T}(X)$, which is also a unitary equivalence preserver. This map is bijective. This can be seen by applying Lemma 4 to $\widehat{\mathcal{T}}$, which again leads to two cases. But the first case $\ker\left(\widehat{\mathcal{T}}\right)\supseteq\ker(\mathrm{tr})\cap\mathfrak{H}_n$ contradicts $\ker(\mathcal{T})=\mathbb{R}I$. The remaining case $\ker\left(\widehat{\mathcal{T}}\right)\subseteq\mathbb{R}I$ leads to $\ker\left(\widehat{\mathcal{T}}\right)=\{0\}$, as $\ker(\mathcal{T})=\mathbb{R}I$ by assumption. Therefore we can restrict ourselves to the case where $\mathcal{T}$ is bijective without loss of generality.

By the unitary equivalence property and bijectivity we have $\mathcal{T}(I)=aI$ for some $a\in\mathbb{R}\setminus\{0\}$. This can be seen by considering $\mathcal{T}(X)=I$, which leads to $\mathcal{T}\left(UXU^\dagger\right)=I$ for all unitaries U, by the preserver property. Using bijectivity of $\mathcal{T}$ this means that $UXU^\dagger=X$ for all unitaries U, which is equivalent to $X=\frac{1}{a}I$ for some $a\in\mathbb{R}\setminus\{0\}$. After rescaling we can assume, without loss of generality, that $\mathcal{T}(I)=I$.

We denote by $\mathfrak{U}_X:=\{B\in\mathfrak{H}_n:B\sim_U X\}$ the unitary orbit at an arbitrary point $X\in\mathfrak{H}_n$. This is a smooth submanifold in $\mathfrak{H}_n=\mathbb{R}^{n^2}$ and therefore we can compute the tangent space at X, see Appendix C,

$$\mathfrak{T}_X\mathfrak{U}_X=\left\{[X,Y]:Y\in\mathfrak{M}_n,Y^\dagger=-Y\right\}.$$

The basic idea of the proof is to use a dimensional argument for the tangent space in order to reduce the unitary equivalence preserver to the rank-1 preserver, which we already understand.

For this argument we compute the dimension of the tangent space at an arbitrary point $X\in\mathfrak{H}_n$. We show that the minimal dimension is attained for X iff $\mathrm{rank}(X-aI)=1$ for some $a\in\mathbb{R}$. This will be used to obtain a rank-1 preserver from the unitary equivalence preserver.

By identifying matrices with vectors, we can compute the real dimension of the tangent space as

$$\dim\left(\mathfrak{T}_X\mathfrak{U}_X\right) = \dim\left(\mathrm{Im}\left([X,\bullet]\right)\right) = \mathrm{rank}\left(X\otimes I - I\otimes X^T\right) ,$$

where we used the representation of the map $Y \mapsto [X,Y]$ by the matrix $X\otimes I - I\otimes X^T \in \mathfrak{M}_n^{\otimes 2}$. As X is hermitian, we can use the spectral theorem to compute the occurring rank exactly. We get

$$\mathrm{rank}\left(X\otimes I - I\otimes X^T\right) = \sharp\{(i,j)\in[n]\times[n] : \lambda_i \neq \lambda_j\} ,$$

where (λ_i) denotes the spectrum of X. It is easy to see, that the minimum is attained for X iff $\mathrm{rank}\left(X - aI\right) = 1$ for some $a\in\mathbb{R}$. Collecting all this together we get for $X \notin \mathbb{R}I$

$$\dim\left(\mathfrak{T}_X\mathfrak{U}_X\right) \geq 2n-2 ,$$

with equality iff $\mathrm{rank}\left(X - aI\right) = 1$ for some $a\in\mathbb{R}$. Note that this includes the case $a = 0$, where X is a rank-1 idempotent.

Now we use that $\mathcal{T}$ preserves unitary equivalence, which implies $\mathcal{T}\left(\mathfrak{U}_X\right) \subseteq \mathfrak{U}_{\mathcal{T}(X)}$. Furthermore as $\mathcal{T}$ is linear its differential coincides with $\mathcal{T}$ itself and by using, what is called push-forward in differential geometry, we get $\mathcal{T}\left(\mathfrak{T}_X\mathfrak{U}_X\right) \subseteq \mathfrak{T}_{\mathcal{T}(X)}\mathfrak{U}_{\mathcal{T}(X)}$.

We will apply this to the special case of $Y\in\mathfrak{H}_n$ such that $\mathcal{T}\left(Y\right) = |y\rangle\langle y|$ for some normalized vector $|y\rangle\in\mathbb{C}^n$, which exists by bijectivity of $\mathcal{T}$. Note that all rank-1 projectors are unitarily equivalent and therefore the concrete $|y\rangle$ does not matter. By the above argumentation we get

$$\mathcal{T}\left(\mathfrak{T}_Y\mathfrak{U}_Y\right) \subseteq \mathfrak{T}_{|y\rangle\langle y|}\mathfrak{U}_{|y\rangle\langle y|} .$$

Considering the dimensions of the spaces involved leads to

$$\dim\left(\mathfrak{T}_Y\mathfrak{U}_Y\right) \leq \dim\left(\mathfrak{T}_{|y\rangle\langle y|}\mathfrak{U}_{|y\rangle\langle y|}\right) = 2n-2 .$$

But by the minimality result, that we obtained above, we get $\dim\left(\mathfrak{T}_Y\mathfrak{U}_Y\right) = 2n-2$ and thus $\mathrm{rank}\left(Y - aI\right) = 1$ for some $a\in\mathbb{R}$, where we used that $\mathcal{T}$ is bijective and that thus the tangent space at $Y\neq 0$ is non-degenerated.

From this we get the existence of a rank-1 projection $|x\rangle\langle x|$ such that $Y = a|x\rangle\langle x| + bI$ and conclude, that the map $\widehat{\mathcal{T}} : \mathfrak{H}_n \to \mathfrak{H}_n$, defined by

$$\widehat{\mathcal{T}}\left(X\right) := a\mathcal{T}\left(X\right) + b\,\mathrm{tr}\left(X\right)I$$

is unitary equivalence preserving and fulfills $\widehat{\mathcal{T}}\left(|x\rangle\langle x|\right) = |y\rangle\langle y|$. As all rank-1 projectors are unitary equivalent this means that $\widehat{\mathcal{T}}$ is a rank-1 preserver on $\mathfrak{H}_n$.

From spectral decomposition we get, that $\widehat{\mathcal{T}}$ is also hermiticity preserving and therefore we can apply Theorem 5 together with the unitary equivalence preserving property to get the existence of a unitary matrix U, such that either $\widehat{\mathcal{T}}(X) = UXU^\dagger$ or $\widehat{\mathcal{T}}(X) = UX^TU^\dagger$ which concludes the proof.

□

Remark 9. *Note that by Lemma 4 the maps in the second case of the above theorem are bijective if and only if*

$$-\frac{bn}{a} \neq 1\,,$$

because a linear, unitary equivalence preserving map $\mathcal{T}$ is bijective if and only if $\mathcal{T}(I_n) \neq 0$.

The special case of quantum channels, which preserve unitary equivalence is of great importance for quantum information theory. We want to call these maps **Werner-Channels**, because their Choi-matrices are the **Werner-states** [Wer89] given by

$$\rho_{\mathrm{W}}(a) := \frac{1}{n^2 - an}\left(I_n \otimes I_n - a\frac{\mathbb{F}_n}{n}\right)$$

for $a \in [-1, 1]$ and the **Werner-Popescu-states** [MH99] given by

$$\rho_{\mathrm{WP}}(a) := \frac{1-a}{n^2} I_n \otimes I_n + a\frac{\omega_n}{n}$$

for $a \in \left[\frac{-1}{n^2-1}, 1\right]$.

These states are uniquely defined, by their invariance under certain local unitary transformations, and the above theorem can be seen as a channel analogue to them.

Werner- and Werner-Popescu-states are of great importance for the question whether their exist NPT-bound entangled states. In fact one can show, that if there are NPT-bound entangled states, then there are also NPT-bound entangled Werner-Popescu states [MH99].

Note also that by Example 2 and the comments to this Example, the Werner channels can be implemented via teleportation with a probability of $p_s = 1$.

We have seen that linear preserver problems provide tools, which can be used to characterize matrix maps in certain cases. In the following discussion, we will use the results proved in this chapter to exclude some properties, that would lead to similar bounds of distillability and quantum capacity as the transposition bounds. These properties are too strong, as they already fix the map to be of a particular trivial form, as the maps characterized in this chapter.

2.2. Convex Analysis and Mapping Cones

In this section we will introduce the theory of mapping cones developed by Størmer et al. [Stø09] [Stø10] [Stø11] [LS] [Sko11] [LS12]. It can be used to better understand the relationship between the various kinds of positive maps introduced in Chapter 1 and to characterize these classes via a duality construction.

In our presentation we will mostly follow [Sko11], but we will use a different notation in order to simplify the theory.

Before introducing the theory, we will start with an important property of the Choi-Jamiolkowski-Isomorphism.

Lemma 5. *Let $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ be a map, then we have*

$$(id_n \otimes \mathcal{T})(\omega_n) = (\vartheta_n \circ \mathcal{T}^* \circ \vartheta_m \otimes id_m)(\omega_m) \ ,$$

where ω_n, ω_m denote maximally entangled states in n,m dimensions and $\mathcal{T}^*$ the adjoint with respect to the Hilbert-Schmidt scalar product. Note that this connects the two possibilities to define the Choi-Matrix for the maps $\mathcal{T}$ and $\vartheta_n \circ \mathcal{T}^* \circ \vartheta_m$.

Proof. By using the elementary properties

$$\begin{aligned}
\operatorname{tr}\left((\mathrm{id}_n \otimes \mathcal{T})(\omega_n) A \otimes B\right) &= \operatorname{tr}\left(\mathcal{T}\left(A^T\right) B\right) \quad \forall A \in \mathfrak{H}_n \forall B \in H_m \\
\operatorname{tr}\left((\mathcal{T} \otimes \mathrm{id}_m)(\omega_m) A \otimes B\right) &= \operatorname{tr}\left(\mathcal{T}\left(B^T\right) A\right) \quad \forall A \in H_m \forall B \in \mathfrak{H}_n
\end{aligned}$$

of the Choi-Jamiolkowski isomorphism, we get

$$\begin{aligned}
\operatorname{tr}\left((\mathrm{id}_n \otimes \mathcal{T})(\omega_n) A \otimes B\right) &= \operatorname{tr}\left(\mathcal{T}\left(A^T\right) B\right) \\
&= \operatorname{tr}\left(\mathcal{T}^*(B) A^T\right) \\
&= \operatorname{tr}\left(\vartheta_n \circ \mathcal{T}^*(B) A\right) \\
&= \operatorname{tr}\left(\vartheta_n \circ \mathcal{T}^* \circ \vartheta_m\left(B^T\right) A\right) \\
&= \operatorname{tr}\left((\vartheta_n \circ \mathcal{T}^* \circ \vartheta_m \otimes \mathrm{id}_m)(\omega) A \otimes B\right)
\end{aligned}$$

for all $A \in \mathfrak{H}_n$ and $B \in \mathfrak{H}_m$ and this finishes the proof.

□

The basic idea of the theory is to consider closed convex cones of positive maps, which are invariant under some kind of local coding. More precise we define the following:

Definition 17. (*Mcs-Cones, symmetric Mcs-Cones. [Sko11]*)

*A non-empty, non-zero, closed convex cone $\mathfrak{C} \subseteq \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$ is called a **cone with a mapping cone symmetry** or short an **mcs-cone** if it has the following property:*

$$\mathcal{T} \in \mathfrak{C} \Rightarrow \forall \mathcal{R} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n), \forall \mathcal{S} \in \mathfrak{CP}(\mathfrak{M}_m, \mathfrak{M}_m) : \mathcal{S} \circ \mathcal{T} \circ \mathcal{R} \in \mathfrak{C}.$$

*For $n = m$ we call an mcs-cone $\mathfrak{C}$ **symmetric** if $\vartheta \mathcal{R}^* \vartheta \in \mathfrak{C}$ whenever $\mathcal{R} \in \mathfrak{C}$.*

Note that by non-zero we mean, that the cone shall not only contain the zero map $\mathfrak{M}_n \ni X \mapsto 0 \in \mathfrak{M}_m$

Example 3. *The following sets of positive maps are mapping cones [Sko11]:*

1. *The sets $\mathfrak{P}_k(\mathfrak{M}_n, \mathfrak{M}_m)$ of k-positive maps.*
2. *The sets $\mathfrak{SP}_k(\mathfrak{M}_n, \mathfrak{M}_m)$ of k-superpositive maps.*
3. *The set $\mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_m)$ of completely positive maps.*
4. *Every set build from the above ones, by applying the operations $\cap, \vee$ and by applying the transposition to the whole set. These cones are called **typical mcs-cones**.*
5. *For $n = m$ all the typical mcs-cones are symmetric.*

It is not yet known, whether there are untypical mcs-cones. This problem seems to be connected to the special properties of the transposition. We will prove later, that this is the only invertible map $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_n$, except the identity, for which the set

$$\mathfrak{C}_{\mathcal{M}} = \{\mathcal{M} \circ \mathcal{R} : \mathcal{R} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)\}$$

is an mcs-cone, namely the completely co-positive maps.

The advantage of the above definition is, that we obtain a new kind of duality. This makes it easier to think about the different sets of positive maps. To introduce this duality, we have to define a scalar product on the set of maps between matrix spaces.

Definition 18. (*Scalar Product for Matrix Maps. [Sko11]*)

For maps $\mathcal{S}, \mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ we set

$$\langle \mathcal{S}, \mathcal{T} \rangle = tr\left(C_{\mathcal{S}}^{\dagger} C_{\mathcal{T}}\right),$$

where $C_{\mathcal{S}}, C_{\mathcal{T}} \mathfrak{M}_{nm}$ denote the Choi-matrices of the maps $\mathcal{S}$ and $\mathcal{T}$.

By linearity of the trace, the properties of the Hilbert-Schmidt scalar-product and the Choi-Jamiolkowski Isomorphism the above map indeed defines a scalar-product. When dealing with hermitian matrices, which is always the case for mcs-cones as they contain only positive and therefore hermitian-preserving maps, we will omit the † in the scalar products.

Note that there are many ways of rewriting the above scalar-product using Lemma 5 and the adjoints of the involved maps. Such rewritings will be heavily used in the following.

With the scalar product defined we can state the following definition.

Definition 19. (*Dual cone. [Sko11]*)
For a non-empty closed convex cone $\mathfrak{C} \subseteq \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$*, we define the* ***dual cone*** *by*

$$\mathfrak{C}^\circ = \{\mathcal{S} \in \mathfrak{H}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m) : \langle \mathcal{S}, \mathcal{T} \rangle \geq 0 \, \forall \mathcal{T} \in \mathfrak{C}\} .$$

The different types of positive maps defined before can be connected using this duality:

Example 4. *It is easy to verify that we have:*

1. $\mathfrak{P}_k(\mathfrak{M}_n, \mathfrak{M}_m)^\circ = \mathfrak{S}\mathfrak{P}_k(\mathfrak{M}_n, \mathfrak{M}_m)$.
2. $\mathfrak{C}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)^\circ = \mathfrak{C}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$.
3. $(\mathfrak{C}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m) \vee \mathfrak{co}\mathfrak{C}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m))^\circ = \mathfrak{C}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m) \cap \mathfrak{co}\mathfrak{C}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$.

An important Theorem is the following:

Theorem 8. (*See [Sko11]*)
Let $\mathfrak{C} \subseteq \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$ *be an mcs-cone, then also* $\mathfrak{C}^\circ$ *is an mcs-cone.*

Proof. By a simple calculation we get

$$\langle \mathcal{R} \circ \mathcal{S} \circ \mathcal{K}, \mathcal{T} \rangle = \operatorname{tr}(C_{\mathcal{R}\circ\mathcal{S}\circ\mathcal{K}} C_{\mathcal{T}}) = \operatorname{tr}(C_{\mathcal{S}} C_{\mathcal{R}^*\circ\mathcal{T}\circ\mathcal{K}^*}) = \langle \mathcal{S}, \mathcal{R}^* \circ \mathcal{T} \circ K^* \rangle \geq 0$$

for all $\mathcal{R} \in \mathfrak{C}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_n)$ and for all $\mathcal{S} \in \mathfrak{C}\mathfrak{P}(\mathfrak{M}_m, \mathfrak{M}_m)$. Here we used Lemma 5 to move the map $\mathcal{K}$ to the other side of the tensor product and then by taking the adjoint to the other side of the scalar product. Then using Lemma 5 we moved it to the correct side of the tensor product. The map $\mathcal{R}$ can be moved directly by taking the adjoint and used that $\mathcal{R} \in \mathfrak{C}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_n) \Leftrightarrow \mathcal{R}^* \in \mathfrak{C}\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_n)$. Therefore we have the mcs-symmetry also in the dual cone.

Convexity and closedness are clearly transferred to the dual cone. It remains to show, that the dual cone is contained in the set $\mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$ of positive maps. For $\mathcal{T} \in \mathfrak{C}$ define the following completely positive maps

$$\mathcal{R}(X) = \langle u| X |u\rangle Y$$
$$\mathcal{K}(X) = \frac{\operatorname{tr}(X)}{\operatorname{tr}(\mathcal{T}(Y))} |v\rangle \langle v|$$

for arbitrary vectors $|u\rangle \in \mathbb{C}_n, |v\rangle \in \mathbb{C}_m$ and $Y \in \mathfrak{M}_n$ such that $\operatorname{tr}(\mathcal{T}(Y)) \neq 0$. We obtain

$$0 \leq \operatorname{tr}(C_{\mathcal{S}} C_{\mathcal{K}\circ\mathcal{T}\circ\mathcal{R}}) = \operatorname{tr}(C_{\mathcal{S}} |u\rangle\langle u| \otimes |v\rangle\langle v|) \ .$$

This shows that $C_{\mathcal{S}}$ is block-positive, which implies that $\mathcal{S}$ is positive, by Corollary 2.

□

The well-known bidual-theorem can be stated in this setting:

Theorem 9. (*See [Sko11]*)
For a closed convex cone $\mathfrak{C} \subseteq \mathfrak{B}(\mathfrak{M}_n, \mathfrak{M}_m)$ *we have*

$$\mathfrak{C}^{\circ\circ} = \mathfrak{C}\,.$$

Proof. See [Roc97]

□

The following theorem provides a characterization of a cone through its dual cone and is of importance for our argumentation.

Theorem 10. (*See [Sko11]*)
For an mcs-cone $\mathfrak{C} \subseteq \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$ *the following are equivalent:*

1. $\mathcal{T} \in \mathfrak{C}$.
2. $\mathcal{S}^* \circ \mathcal{T} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)\ \forall \mathcal{S} \in \mathfrak{C}^\circ$.
3. $\mathcal{T} \circ \mathcal{S}^* \in \mathfrak{CP}(\mathfrak{M}_m, \mathfrak{M}_m)\ \forall \mathcal{S} \in \mathfrak{C}^\circ$.

Proof. We will only prove the equivalence of 1. and 2., as the equivalence of 1. and 3. works in analogy.
1. $\Rightarrow$ **2.** :
For $\mathcal{T} \in \mathfrak{C}$ we have $\mathcal{T} \circ \mathcal{R} \in \mathfrak{C}$ for all $\mathcal{R} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$. Now take $\mathcal{S} \in \mathfrak{C}^\circ$ and by Lemma 5 together with the adjoint operation we can derive

$$\begin{aligned} 0 \leq \operatorname{tr}(C_{\mathcal{S}} C_{\mathcal{T}\circ\mathcal{R}}) &= \operatorname{tr}(C_{\mathcal{S}} (\vartheta_n \circ \mathcal{R}^* \circ \vartheta_n \otimes \mathrm{id}_m)(C_{\mathcal{T}})) \\ &= \operatorname{tr}(\omega_n (\vartheta_n \circ \mathcal{R}^* \circ \vartheta_n \otimes \mathrm{id}_n)(C_{\mathcal{S}^*\circ\mathcal{T}})) \\ &= \operatorname{tr}((\vartheta_n \circ \mathcal{R} \circ \vartheta_n \otimes \mathrm{id}_n)(\omega_n)\, C_{\mathcal{S}^*\circ\mathcal{T}}) \end{aligned}$$

for all $\mathcal{R} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$. This is equivalent to $\mathcal{S}^* \circ \mathcal{T} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$.

2. $\Rightarrow$ **1.** :

Let $\mathcal{S}^* \circ \mathcal{T} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$ $\forall \mathcal{S} \in \mathfrak{C}^\circ$. As $C_{\mathcal{S}^* \circ \mathcal{T}} \geq 0$ by assumption we get

$$0 \leq \mathrm{tr}\left(\omega_n C_{\mathcal{S}^* \circ \mathcal{T}}\right) = \mathrm{tr}\left(C_{\mathcal{S}} C_{\mathcal{T}}\right) \ \forall \mathcal{S} \in \mathfrak{C}^\circ .$$

This shows $\mathcal{T} \in \mathfrak{C}^{\circ\circ} = \mathfrak{C}$, where we used Theorem 9.

□

Note that all the typical mcs-cones introduced in Example 3 are symmetric, which means that if $\mathcal{T} \in \mathfrak{C}$ we have also $\mathcal{T}^* \in \mathfrak{C}$. Therefore we can omit the adjoints in the above theorem, when dealing with typical mcs-cones.

Example 5. *By Example 4 this theorem allows the characterization of many important classes of maps. For example $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ is decomposable iff*

$$\mathcal{S} \circ \mathcal{T} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n) \ \forall \mathcal{S} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_m) \cap \mathfrak{coCP}(\mathfrak{M}_n, \mathfrak{M}_m) \ .$$

This shows a connection between decomposable maps and maps that are both completely positive and completely co-positive.

As another example we get, that $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ is positive iff

$$\mathcal{T} \circ \mathcal{S} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n) \ \forall \mathcal{S} \in \mathfrak{SP}_1(\mathfrak{M}_n, \mathfrak{M}_m) \ ,$$

which gives a characterization of positive maps via entanglement breaking maps. As the entanglement breaking maps correspond to the separable states by the Choi-Jamiolkowski Isomorphism this is equivalent to the characterization of positive maps in Corollary 2.

We have introduced the formalism of mapping cones and proved some theorems characterizing such cones, which we will use later in our discussion.

3. Entanglement Distillation

In this chapter we will study the first problem defined in Chapter 1, namely to find criteria for a quantum state $X \in \mathfrak{B}\left(\mathbb{C}^{m_A} \otimes \mathbb{C}^{m_B}\right)$ on a bipartite system to be not distillable.

We showed that the transposition map leads to such a criterion and here we study the possibility of obtaining a similar criterion from different maps. Assuming certain properties used in the proof for the transposition criterion, we will prove that they already fix the map fulfilling them to be the transposition. However we were not able to exclude the possibility of another criterion in a broader setting.

Repeating a result from the introduction we will start with a different proof for the transposition criterion. This will make it easier to think about generalizations of the theorem. In the proof we will use the following Lemma

Lemma 6. *Let $\mathcal{S} \in \mathfrak{Sep}\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B}\right)$ be given. Then we get*

$$\mathcal{S}' = \left(id_{m_A} \otimes \vartheta_{m_B}\right) \circ \mathcal{S} \circ \left(id_{n_A} \otimes \vartheta_{n_B}\right) \in \mathfrak{Sep}\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B}\right) \ ,$$

i.e. the partial transposition applied in the above way, preserves separability of the maps involved.

Proof. Take the Kraus-decomposition, see Appendix B,

$$\mathcal{S}\left(Y\right) = \sum_{i=1}^{N} \left(\left(A_i \otimes B_i\right) X \left(A_i \otimes B_i\right)^\dagger\right) \ .$$

A simple calculation, using $\left(AB\right)^T = B^T A^T$ for all $A, B \in \mathfrak{M}_n$, shows

$$\mathcal{S}'\left(Y\right) = \sum_{i=1}^{N} \left(\left(A_i \otimes \overline{B_i}\right) Y \left(A_i^\dagger \otimes B_i^T\right)\right) \ ,$$

which leads to $\mathcal{S}' \in \mathfrak{Sep}\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}, \mathfrak{M}_{m_A} \otimes \mathfrak{M}_{m_B}\right)$.

□

In the proof of the transposition criterion we will use the above lemma to exchange the transposition with a separable map, leaving behind a different separable map, which is in particular completely positive. This is the important part of the proof and any map, with this property would lead to a similar criterion.

Theorem 11. (*See [MH98]*)
If for a quantum state $X \in \mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}$

$$X^{T_B} \geq 0$$

is fulfilled, then X is not distillable.

Proof. Assume that X is distillable and $X^{T_B} \geq 0$. Then by Definition 13 and for $\epsilon = \frac{1}{4}$ there exists $k \in \mathbb{N}$ and an $\mathcal{L} \in \mathfrak{Locc}\left(\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}\right)^{\otimes k}, \mathfrak{M}_2 \otimes \mathfrak{M}_2 \right)$ such that

$$\|\frac{\omega_2}{2} - \mathcal{L}\left(X^{\otimes k}\right)\|_1 \leq \frac{1}{4} \, .$$

By applying the partial transposition to this inequality and by using $\|\mathrm{id}_2 \otimes \vartheta_2\|_{1\to 1} = 2$ we get

$$\|\frac{\mathbb{F}_2}{2} - \mathcal{L}'\left(\left(X^{T_B}\right)^{\otimes k} \right)\|_1 = \|\frac{\omega_2^{T_B}}{2} - \mathcal{L}\left(X^{\otimes k}\right)^{T_B}\|_1 \leq \frac{1}{2} \, ,$$

for the map $\mathcal{L}' \in \mathfrak{Sep}\left(\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}\right)^{\otimes k}, \mathfrak{M}_2 \otimes \mathfrak{M}_2 \right)$ given by

$$\mathcal{L}' = \left(\mathrm{id}_2 \otimes \vartheta_2\right) \circ \mathcal{L} \circ \left(\mathrm{id}_{kn_A} \otimes \vartheta_{kn_B}\right) \, .$$

From Lemma 6 we get that $\mathcal{L}' \in \mathfrak{Sep}\left(\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}\right)^{\otimes k}, \mathfrak{M}_2 \otimes \mathfrak{M}_2 \right)$ which implies $\|\mathcal{L}'\left(\left(X^{T_B}\right)^{\otimes k} \right)\|_1 \leq \|\mathcal{L}'\|_{1\to 1} \leq 1$ as $\mathcal{L}'$ is in particular trace-preserving and completely positive and X a quantum state, see Appendix A for details about the norms.

Using the triangle inequality and $\|\frac{\mathbb{F}_2}{2}\|_1 = 2$, we get

$$1 \leq \|\frac{\mathbb{F}_2}{2} - \mathcal{L}'\left(\left(X^{T_B}\right)^{\otimes k} \right)\|_1 \, ,$$

which gives a contradiction to the inequality derived above.

□

We have to define, what we mean by a map to generate a distillability criterion. Note that we do not search for a quantitative bound on the distillability, but for a criterion in the most simple case.

Definition 20. (*Generalized Peres-Horodecki-Criterion*)
*A map $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_n$ is said to **generate a distillability criterion**, if for all quantum states $X \in \mathfrak{M}_k \otimes \mathfrak{M}_n$*

$$\left(id_k \otimes \mathcal{M}\right)(X) \geq 0 \Longrightarrow X \text{ is not distillable.}$$

for some $k \in \mathbb{N}$.

The transposition generates a distillability criterion. As an easy necessary condition for an arbitrary map $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_n$ to generate a distillability criterion, we have that $\mathcal{M} \notin \mathfrak{P}_k(\mathfrak{M}_n, \mathfrak{M}_n)$ for $k \geq 2$. Because otherwise we would have $(\mathrm{id}_k \otimes \mathcal{M})\left(\frac{\omega_k}{k}\right) \geq 0$, which would give a contradiction as the state $\frac{\omega_k}{k}$ is trivially distillable. In particular a completely positive map cannot generate a distillability criterion.

Note that if the map $\mathcal{M} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_n)$ generates a distillability criterion, it detects all separable states $X \in \mathfrak{M}_k \otimes \mathfrak{M}_n$ for all $k \in \mathbb{N}$ as not distillable. This is of course correct. As we have not used positivity of the transposition in the above proof, but obtained a criterion for the case $X^{T_B} \geq 0$, this is an optional property.

In this chapter we will only consider hermiticity preserving maps generating a distillability criterion. The reason for this restriction is, that arbitrary non-positive maps are much more difficult to handle than hermiticity preserving ones. But in principle such maps could lead to interesting criteria. We will prove some results for more general maps in the following chapter.

In order to derive maps which generate a distillability criterion let us see which properties of the transposition were needed in the above proof.

The most important property that we used, is multiplicativity or antimultiplicativity together with linearity.

Definition 21. *A map $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_m$ is called*

- ***multiplicative****, if $\mathcal{M}(AB) = \mathcal{M}(A)\mathcal{M}(B)$,*
- ***antimultiplicative****, if $\mathcal{M}(AB) = \mathcal{M}(B)\mathcal{M}(A)$,*

for all $A, B \in \mathfrak{M}_n$.

The transposition is of course antimultiplicative and linear which was important for the proof of Lemma 6.

There are however not many maps, which are positive, linear and multiplicative or antimultiplicative. For bijective maps this can be seen quite easily by the fact, that such a map would be a Jordan Automorphism and therefore we know that there are just two possibilities, namely the identity and the transposition, up to unitary similarity.

Our argumentation will be very elementary and uses linear preserver results just at the most final point.

Let us start with some lemmata:

Lemma 7. *Let $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_m$ be multiplicative or antimultiplicative and such that there exists an invertible $X \in Im(\mathcal{M})$, then $\mathcal{M}(I_n) = I_m$.*

Proof. Trivial.

□

In the following Lemma we characterize an inverse of a multiplicative or antimultiplicative map restricted to its image.

Lemma 8. *Let $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_m$ be multiplicative or antimultiplicative and such that there exists an invertible $X \in Im(\mathcal{M})$ then we have $\frac{n}{m}\mathcal{M}^*\left(\mathcal{M}(A)^\dagger\right)^\dagger = A$, i.e. the map $\frac{n}{m}\mathcal{H} \circ \mathcal{M}^* \circ \mathcal{H} : \mathfrak{M}_m \to \mathfrak{M}_n$ inverts $\mathcal{M}$ on $Im(\mathcal{M})$, where we set $\mathcal{H}(X) = X^\dagger$.*

Proof. For a multiplicative map we have:

$$\begin{aligned}
\operatorname{tr}\left(A^\dagger \mathcal{M}^*(C) B^\dagger\right) &= \operatorname{tr}\left((AB)^\dagger \mathcal{M}^*(C)\right) \\
&= \operatorname{tr}\left(\mathcal{M}(AB)^\dagger C\right) \\
&= \operatorname{tr}\left([\mathcal{M}(A)\mathcal{M}(B)]^\dagger C\right) \\
&= \operatorname{tr}\left(\mathcal{M}(A)^\dagger C \mathcal{M}(B)^\dagger\right) \\
&= \operatorname{tr}\left(A^\dagger \mathcal{M}^*\left(C\mathcal{M}(B)^\dagger\right)\right) .
\end{aligned}$$

As this is true for all $A, B \in \mathfrak{M}_n$ and $C \in \mathfrak{M}_m$ we have

$$\mathcal{M}^*(C) B^\dagger = \mathcal{M}^*\left(C\mathcal{M}(B)^\dagger\right) .$$

By doing a similar calculation as above starting with $\operatorname{tr}\left(B^\dagger \mathcal{M}^*(C) A^\dagger\right)$, we also obtain

$$B^\dagger \mathcal{M}^*(C) = \mathcal{M}^*\left(\mathcal{M}(B)^\dagger C\right) . \tag{3.1}$$

The two expressions combined and with $C = I_m$ give

$$\mathcal{M}^*(I_m) B^\dagger = \mathcal{M}^*\left(I_m \mathcal{M}(B)^\dagger\right) = \mathcal{M}^*\left(\mathcal{M}(B)^\dagger I_m\right) = B^\dagger \mathcal{M}^*(I_m) .$$

But this means that $[\mathcal{M}^*(I_m), B] = 0$ for all $B \in \mathfrak{M}_n$ which is equivalent to $\mathcal{M}^*(I_m) = cI_n$ for $c \in \mathbb{C}$. We get from Lemma 7 that

$$m = \operatorname{tr}(I_m \mathcal{M}(I_n)) = \operatorname{tr}(\mathcal{M}^*(I_m) I_n) = \operatorname{tr}(cI_n) = cn$$

and thus $c = \frac{m}{n}$. Plugging this into Equation 3.1 for $C = I_m$ we get

$$A = \frac{n}{m}\mathcal{M}^*\left(\mathcal{M}(A)^\dagger\right)^\dagger ,$$

The antimultiplicative case works in an analogy.

□

For a bijective $\mathcal{M} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_n)$ this would allow a characterization of $\mathcal{M}$ via Corollary 5. In this case we would have $c \in \mathbb{R}^+$ and $\mathcal{M}^* \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_n)$. The Lemma would provide a positive inverse of $\mathcal{M}$, which finishes the proof using the corollary.

The following corollary can be derived in the surjective case:

Corollary 7. *Let $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_n$ be multiplicative or antimultiplicative and surjective, then $\mathcal{M}$ is trace-preserving.*

Proof. Follows immediately from Lemma 8 and Lemma 7, as $\mathcal{M}(I_n) = I_n$.

□

We will go on with our elementary reasoning for $n \neq m$ now.

The next lemma will be used to characterize the Choi-Matrices that correspond to multiplicative or antimultiplicative maps.

Lemma 9. *For $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_m$ and such that there exists an invertible $X \in Im(\mathcal{M})$ we have*

- *If $\mathcal{M}$ is multiplicative, then $C_{\mathcal{M}}^2 = nC_{\mathcal{M}}$.*
- *If $\mathcal{M}$ is antimultiplicative, then $C_{\mathcal{M}}^2 = I_{nm}$.*

Proof. For an arbitrary $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_m$ we have

$$\begin{aligned} C_{\mathcal{M}}^2 &= \sum_{ij} |i\rangle\langle j| \otimes \mathcal{M}(|i\rangle\langle j|) \sum_{kl} |k\rangle\langle l| \otimes \mathcal{M}(|k\rangle\langle l|) \\ &= \sum_{i=1,l=1,k=1}^{n} |i\rangle\langle l| \otimes \mathcal{M}(|i\rangle\langle k|)\,\mathcal{M}(|k\rangle\langle l|) \ . \end{aligned}$$

If $\mathcal{M}$ is multiplicative we have

$$\begin{aligned} C_{\mathcal{M}}^2 &= \sum_{ilk} |i\rangle\langle l| \otimes \mathcal{M}(|i\rangle\langle l|) \\ &= nC_{\mathcal{M}} \ . \end{aligned}$$

If $\mathcal{M}$ is antimultiplicative we have

$$\begin{aligned} C_{\mathcal{M}}^2 &= \sum_{ik} |i\rangle\langle i| \otimes \mathcal{M}(|k\rangle\langle k|) \\ &= I_n \otimes \mathcal{M}(I_n) = I_{nm} \ , \end{aligned}$$

where we used Lemma 7.

□

With this lemmata we get the following characterization

Theorem 12. *For $\mathcal{M} \in \mathfrak{HP}(\mathfrak{M}_n, \mathfrak{M}_m)$ and such that there exists an invertible $X \in Im(\mathcal{M})$ we have*

- *If $\mathcal{M}$ is multiplicative, then we have $\mathcal{M} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_m)$.*
- *If $\mathcal{M}$ is antimultiplicative, then we have $\mathcal{M} \in \mathfrak{coCP}(\mathfrak{M}_n, \mathfrak{M}_m)$.*

Proof. As $\mathcal{M} \in \mathfrak{HP}(\mathfrak{M}_n, \mathfrak{M}_m)$ the Choi matrix $C_{\mathcal{M}}$ is hermitian, by Corollary 2. When $\mathcal{M}$ is M we find by Lemma 9 that

$$C_{\mathcal{M}}^{\dagger} C_{\mathcal{M}} = n C_{\mathcal{M}}$$

which is only possible if $C_{\mathcal{M}} \geq 0$.

In the case when $\mathcal{M}$ is antimultiplicative we know that $\vartheta_n \circ \mathcal{M}$ is a positive, linear and multiplicative map and therefore completely positive. But this means that $\mathcal{M} \in \mathfrak{coCP}(\mathfrak{M}_n, \mathfrak{M}_m)$. □

Thus a hermitian preserving multiplicative map has to be completely positive and cannot be used to generate a distillation criterion. Only an antimultiplicative hermitian preserving map could generate such a criterion, but as the map has to be completely co-positive in this case, the criterion would be strongly connected to the ordinary transposition criterion.

In the antimultiplicative case we have the following theorem for $n = m$:

Theorem 13. *For $\mathcal{M} \in \mathfrak{HP}(\mathfrak{M}_n, \mathfrak{M}_n)$ bijective and antimultiplicative, there exists an $U \in \mathfrak{U}_n$ such that*

$$\mathcal{M}(X) = U X^T U^{\dagger}.$$

Proof. By Theorem 12 we know $\mathcal{M}$ to be completely co-positive and therefore in particular positive. As $n = m$, we get by Lemma 8 that $\mathcal{M}^{-1} = \mathcal{H} \circ \mathcal{M}^* \circ \mathcal{H}$. As a map is positive iff its adjoint is positive, we follow from Corollary 5, that there is an invertible matrix $S \in \mathfrak{M}_n$ such that $\mathcal{M}(X) = S X^T S^{\dagger}$. From Lemma 7 we get that S has to be unitary and we are done. □

Remark 10. *This could have also been proven by applying Wigner's Theorem, Corollary 6. A trace-preserving multiplicative or antimultiplicative map $\mathcal{M} \in \mathfrak{HP}(\mathfrak{M}_n, \mathfrak{M}_n)$ fulfills*

$$tr(XY) = tr(\mathcal{M}(XY)) = tr(\mathcal{M}(X)\mathcal{M}(Y))$$

for all $X, Y \in \mathfrak{M}_n$. By the multiplicativity or antimultiplicativity we also get, that rank-1 idempotents are mapped to rank-1 idempotents. This leads using Wigner's Theorem to the two possibilities $\mathcal{M}(X) = U X U^{\dagger}$ or $\mathcal{M}(X) = U X^T U^{\dagger}$ for unitary U.

The assumption of multiplicativity or antimultiplicativity of a linear hermiticity preserving map fixes it to be either completely positive or completely co-positive. In the cases of completely positive maps, or bijective maps we can conclude, that those maps will not lead to new criteria for distillability in the sense of Definition 20. The remaining cases of completely co-positive maps might provide a new criterion it seems however unlikely, that this can lead to NPT-bound entangled states.

We will provide some evidence for this in the next chapter, where we prove similar results under milder assumptions for the case of quantum capacity bounds. This will lead also to results for distillability criteria, as there is a connection to quantum capacity bounds.

4. Capacity Bounds

In this chapter we will state our results concerning problem 2 from chapter 1, namely trying to generalize the transposition bound for the quantum channel capacity. This is connected to problem 1 of finding distillability criteria.

First we will try to clarify this connection and then show that there are no such generalizations to the known bound, satisfying certain restrictive conditions. In the course of our discussion, we will also show the transposition bound for the two-way quantum capacity $\mathcal{Q}_2$, which we introduced in chapter 1. This result is not unexpected, but has to our knowledge not been published.

4.1. Connection of Capacity Bounds and Entanglement Distillation

We will use this chapter to explain the connection between the capacity bound, Theorem 3, and the distillation criteria from the last chapter.

By the Choi-Jamiolkowski-Isomorphism we have a quantum state $\frac{C_{\mathcal{T}}}{n} \in \mathcal{B}\left(\mathbb{C}^n \otimes \mathbb{C}^m\right)$ corresponding to the quantum channel $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$. Note that this is not true in the other direction. There are of course quantum states, which are not the Choi-Matrix of a quantum channel, but of some completely positive map, which is not trace-preserving. We have the following theorem connecting the maximal rate of distillability $\mathcal{D}_2\left(\frac{C_{\mathcal{T}}}{n}\right)$ of the Choi-Matrix to the two-way quantum capacity $\mathcal{Q}_2\left(\mathcal{T}\right)$.

Theorem 14. *For a quantum channel $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ we have*

1.) $\mathcal{D}_2\left(\frac{C_{\mathcal{T}}}{n}\right) \leq \mathcal{Q}_2\left(\mathcal{T}\right)$.

2.) $\frac{1}{n^2}\mathcal{Q}_2\left(\mathcal{T}\right) \leq \mathcal{D}_2\left(\frac{C_{\mathcal{T}}}{n}\right)$.

Proof. For the first inequality we have to show, that every achievable rate $r \in \mathbb{R}^+$ for the distillation of $\frac{C_{\mathcal{T}}}{n}$ is also an achievable rate for the 2-way assisted quantum communication, see 12.

Let $r \in \mathbb{R}^+$ be an achievable rate for the distillation of the state $\frac{C_{\mathcal{T}}}{n}$, i.e. there exist sequences $\left(k^1_\nu\right)_{\nu\in\mathbb{N}}$ and $\left(k^2_\nu\right)_{\nu\in\mathbb{N}}$ such that $\frac{k^1_\nu}{k^2_\nu} \to r$ and $k^2_\nu \to \infty$ for $\nu \to \infty$ and $\mathcal{L}_\nu \in \mathfrak{Locc}\left(\left(\mathfrak{M}_{n_A} \otimes \mathfrak{M}_{n_B}\right)^{\otimes k^2_\nu}, \left(\mathfrak{M}_2 \otimes \mathfrak{M}_2\right)^{\otimes k^1_\nu}\right)$ such that

$$\| \left(\frac{\omega_2}{2}\right)^{\otimes k^1_\nu} - \mathcal{L}_\nu\left(\left(\frac{C_{\mathcal{T}}}{n}\right)^{\otimes k^2_\nu}\right) \|_1 \to 0$$

for $\nu \to \infty$.

We have to construct a coding scheme

$$\mathcal{C}_{k_\nu^1,k_\nu^2}:\mathfrak{CPTP}\left(\mathfrak{M}_n,\mathfrak{M}_m\right)\to\mathfrak{CPTP}\left(\mathfrak{M}_2^{\otimes k_\nu^2},\mathfrak{M}_2^{\otimes k_\nu^2}\right)$$

of the form

$$\mathcal{C}_{k_\nu^1,k_\nu^2}\left[\mathcal{T}\right]=\mathrm{tr}_W\circ\mathcal{L}_{\mathrm{AB}}^{k_\nu^1+1}\circ\widehat{\mathcal{T}}_{k_\nu^1}\circ\mathcal{L}_{\mathrm{AB}}^{k_\nu^1}\circ\cdots\circ\widehat{\mathcal{T}}_2\circ\mathcal{L}_{\mathrm{AB}}^2\circ\widehat{\mathcal{T}}_1\circ\mathcal{L}_{\mathrm{AB}}^1\circ\mathcal{E}_W\ ,$$

as specified in Definition 12. Note that the distillation protocol, which we assumed, can be implemented by LOCC-maps, and therefore can be used in such a coding scheme.

We choose $m_l^A=k_\nu^2$ and $m_l^B=0$ for $l\leq k_\nu^2$ and we introduce a working Hilbert-spaces at A and B, which consists of k_ν^2 registers, which are spaces $\mathfrak{M}_n$ at A and spaces $\mathfrak{M}_m$ at B, i.e.

$$\mathfrak{h}_W^A=\mathfrak{M}_n^{\otimes k_\nu^2}$$

and

$$\mathfrak{h}_W^B=\mathfrak{M}_m^{\otimes k_\nu^2}$$

initially in states $|0\rangle\langle 0|^{\otimes k_\nu^2}$. We will denote the working Hilbert spaces by the labels W^A and W^B and the single registers by the labels $W_1^A,\ldots,W_{k_\nu^2}^A$ and $W_1^B,\ldots,W_{k_\nu^2}^B$. Take the LOCC-maps

$$\mathcal{L}_{\mathrm{AB}}^k:\mathfrak{h}_W^A\otimes\mathfrak{h}_W^B\otimes\mathfrak{M}_2^{\otimes k_n^1 u}\otimes\mathfrak{M}_m^B\to\mathfrak{h}_W^A\otimes\mathfrak{h}_W^B\otimes\mathfrak{M}_2^{\otimes k_\nu^1}\otimes\mathfrak{M}_n^A\ ,$$

where we denote by $\mathfrak{M}_2^{\otimes k_\nu^1}$ the matrix space containing the state $X^{\otimes k_\nu^1}$, which we want to communicate from A to B. Note that this space is not changed by these first LOCC-maps, which act as

$$\begin{aligned}\mathcal{L}_{AB}^k&\left(|0\rangle\langle 0|_{W^A}^{\otimes\left(k_\nu^2-l\right)}\otimes\left(\frac{C_{\mathcal{T}}^{W^A:W^B}}{n}\right)^{\otimes(l-1)}\otimes|0\rangle\langle 0|_{W^B}^{\otimes\left(k_\nu^2-l+1\right)}\otimes X^{\otimes k_\nu^1}\otimes\frac{C_{\mathcal{T}}^{W^A_{\left(k_\nu^2-l+1\right)}:B}}{n}\right)\\&=|0\rangle\langle 0|_{W^A}^{\otimes\left(k_\nu^2-l-1\right)}\otimes\left(\frac{C_{\mathcal{T}}^{W^A:W^B}}{n}\right)^{\otimes l}\otimes|0\rangle\langle 0|_{W^B}^{\otimes\left(k_\nu^2-l\right)}\otimes X^{\otimes k_\nu^1}\otimes\frac{\omega_n^{W^A_{\left(k_\nu^2-l\right)}:A}}{n}\ ,\end{aligned}$$

when applied together with $\mathcal{T}$ in the way denoted in the definition of the coding scheme $\mathcal{C}_{k_\nu^1,k_\nu^2}\left[\mathcal{T}\right]$. Here we had to reorder the involved Hilbert spaces to take the entangled states into account and the occurring Choi-matrices are shared over the denoted bipartitions.

Applying all the maps defined so far together we obtain

$$\mathcal{L}^f_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_{k^2_\nu} \circ \mathcal{L}^{k^2_\nu}_{\mathrm{AB}} \circ \cdots \circ \widehat{\mathcal{T}}_2 \circ \mathcal{L}^2_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_1 \circ \mathcal{L}^1_{\mathrm{AB}} \circ \mathcal{E}_W\,(X) = \left(\frac{C^{W_A:W_B}_{\mathcal{T}}}{n}\right)^{\otimes k^2_\nu} \otimes X^{\otimes k^1_\nu}\,,$$

where the map $\mathcal{L}^f_{\mathrm{AB}}$ was introduced as a formality to move the last Choi-matrix into the working Hilbert space. Now we apply the distillation map $\mathcal{L}_\nu$ to the normalized Choi-Matrices, which is by construction LOCC, to obtain

$$\begin{aligned}&\left(\mathcal{L}_\nu \otimes \mathrm{id}_2^{\otimes k^1_\nu}\right) \circ \mathcal{L}^f_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_{k^2_\nu} \circ \mathcal{L}^{k^2_\nu}_{\mathrm{AB}} \circ \cdots \circ \widehat{\mathcal{T}}_2 \circ \mathcal{L}^2_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_1 \circ \mathcal{L}^1_{\mathrm{AB}} \circ \mathcal{E}_W\,(X)\\ &= \left(\frac{\omega_2}{2}\right)^{\otimes k^1_\nu} \otimes X^{\otimes k^1_\nu} + Y_\nu\,,\end{aligned}$$

with $Y_\nu \to 0$ for $\nu \to \infty$, to take into account, that the distillation rate is achieved only in the limit $\nu \to \infty$. But now we can apply the LOCC-Map corresponding to quantum teleportation, see Example 2 for details, using the shared entangled states between A and B. This is possible with success probability $p_s = 1$ as many times as there are maximally entangled states left. We see that the total rate of this protocol is the rate $r \in \mathbb{R}^+$ by which we can distill the state $\frac{C_{\mathcal{T}}}{n}$.

For the second inequality let $r \in \mathbb{R}^+$ be an achievable rate for two-way quantum communication. We have to show, that $\frac{r}{n^2}$ is an achievable rate for the distillation of the Choi-matrix.

Therefore take sequences $\left(k^1_\nu\right)_{\nu\in\mathbb{N}}$, $\left(k^2_\nu\right)_{\nu\in\mathbb{N}}$ such that $\frac{k^1_\nu}{k^2_\nu} \to r$ and $k^2_\nu \to \infty$ for $\nu \to \infty$. We are given k^2_ν copies of the Choi-matrix $\frac{C_{\mathcal{T}}}{n}$ on the bipartite Hilbert space $\mathfrak{M}^A_n \otimes \mathfrak{M}^B_m$. First prepare the following state, which is trivially possible via LOCC-Operations with respect to the bipartition (A:B),

$$\left(\frac{\omega_n}{n}\right)^{\otimes k^2_\nu} \otimes C_{\mathcal{T}}^{\otimes k^2_\nu} \in \left(\mathfrak{M}^{A'}_n \otimes \mathfrak{M}^{A''}_n \otimes \mathfrak{M}^A_n \otimes \mathfrak{M}^B_m\right)^{\otimes k^2_\nu}\,.$$

Now take a coding scheme

$$\mathcal{C}_{k^1_\nu,k^2_\nu} : \mathfrak{CPTP}\left(\mathfrak{M}_n, \mathfrak{M}_m\right) \to \mathfrak{CPTP}\left(\mathfrak{M}_2^{\otimes k^2_\nu}, \mathfrak{M}_2^{\otimes k^2_\nu}\right)$$

of rate $r \in \mathbb{R}^+$ for the two-way assisted quantum communication of the form

$$\mathcal{C}_{k^1_\nu,k^2_\nu}\left[\mathcal{T}\right] = \mathrm{tr}_W \circ \mathcal{L}^{k^1_\nu+1}_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_{k^1_\nu} \circ \mathcal{L}^{k^1_\nu}_{\mathrm{AB}} \circ \cdots \circ \widehat{\mathcal{T}}_2 \circ \mathcal{L}^2_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_1 \circ \mathcal{L}^1_{\mathrm{AB}} \circ \mathcal{E}_W\,.$$

We will use this coding scheme, to send one half of the maximally entangled state through the channel, implementing the channel uses via teleportation as explained in Example 2. Therefore first apply the coding scheme up to the first channel use, to one half of the state $\frac{\omega_n}{n} \in \mathfrak{M}^{A'}_n \otimes \mathfrak{M}^{A''}_n$

Now we do a measurement using the projector-valued measure $\{\frac{\omega^A_n}{n}, I^A_n \otimes I^A_n - \frac{\omega^A_n}{n}\}$ to the bipartition $(A'' : A)$, which is of course LOCC with respect to the bipartition (A:B), and thus implementing $\mathcal{T}$ via teleportation with success probability $p_s = \frac{1}{n^2}$. By repeating this for the rest of the coding scheme always implementing $\mathcal{T}$ via teleportation and repeating any attempt, where the teleportation protocol failed, we end up with the state

$$\begin{aligned}&\left(\frac{\omega_2^{A''B}}{2}\right)^{\otimes\widehat{k}_\nu^1}\otimes\left(\frac{\omega_n^{A'A}}{n}\right)^{\otimes\widehat{k}_\nu^1}\otimes Y^{\otimes\left(k_\nu^2-\widehat{k}_\nu^1\right)}+E_\nu\\&\in\left(\mathfrak{M}_2^{A''}\otimes\mathfrak{M}_2^{B}\otimes\mathfrak{M}_n^{A'}\otimes\mathfrak{M}_n^{A}\right)^{\otimes\widehat{k}_\nu^1}\otimes\left(\mathfrak{M}_2^{A''}\otimes\mathfrak{M}_2^{B}\otimes\mathfrak{M}_n^{A'}\otimes\mathfrak{M}_n^{A}\right)^{\otimes\left(k_\nu^2-\widehat{k}_\nu^1\right)}\ ,\end{aligned}$$

where we changed the order of Hilbert spaces for convenience, and used the coding scheme of rate $r\in\mathbb{R}^+$ to send the maximally entangled state to A. We denoted by Y the collection of states where the projector $\left(I_n^A\otimes I_n^A-\frac{\omega_n^A}{n}\right)$ was applied and which were discarded in the protocol. By E_ν we denoted an error state for which we have $E_\nu\to 0$ for $\nu\to\infty$. Finally, by combining the communication rate R with the probability of $\frac{1}{n^2}$ that the right projector was applied, we get $\frac{\widehat{k}_\nu^1}{k_\nu^2}\to\frac{r}{n^2}$.

□

Remark 11. *The constant $\frac{1}{n^2}$ in the second inequality can be improved, by using the channel-dependent projector-valued measure as discussed in Example 2. In the case of unitary equivalence preserving channels, which are the Werner channels by Theorem 4, we have $\mathcal{Q}_2(\mathcal{T})=\mathcal{D}_2\left(\frac{C_\mathcal{T}}{2}\right)$. We do not know whether this is also true in general.*

A simple consequence of the above theorem is

$$\mathcal{Q}_2(\mathcal{T})=0\Leftrightarrow\mathcal{D}_2\left(\frac{C_\mathcal{T}}{n}\right)=0\ ,$$

which connects the two quantities. Channels with zero 2-way quantum capacity give non-distillable states via the Choi-Jamiolkowski isomorphism.

We will now prove the transposition bound for the two-way quantum capacity, which shows, that for states, which are Choi-Matrices of quantum channels the transposition bound for the channel capacity and the transposition criterion are the same.

Theorem 15. *For a quantum channel $\mathcal{T}:\mathfrak{M}_n\to\mathfrak{M}_m$ we have*

$$\mathcal{Q}_2(\mathcal{T})\leq log\left(\|\vartheta\circ\mathcal{T}\|_\diamond\right)\ ,$$

where ϑ denotes the transposition.

Proof. The proof works in analogy to the more special Theorem 3. Let $r\in\mathbb{R}^+$ be an achievable rate for the channel $\mathcal{T}$ and sequences $\left(k_\nu^1\right)_{\nu\in\mathbb{N}}$, $\left(k_\nu^2\right)_{\nu\in\mathbb{N}}$ such that $\frac{k_\nu^2}{k_\nu^1}\to r$ for $\nu\to\infty$. Take a suitable two-way coding scheme

$$\mathcal{C}_{k_\nu^1,k_\nu^2}:\mathfrak{CPTP}\left(\mathfrak{M}_n,\mathfrak{M}_m\right)\to\mathfrak{CPTP}\left(\mathfrak{M}_2^{\otimes k_\nu^2},\mathfrak{M}_2^{\otimes k_\nu^2}\right)$$

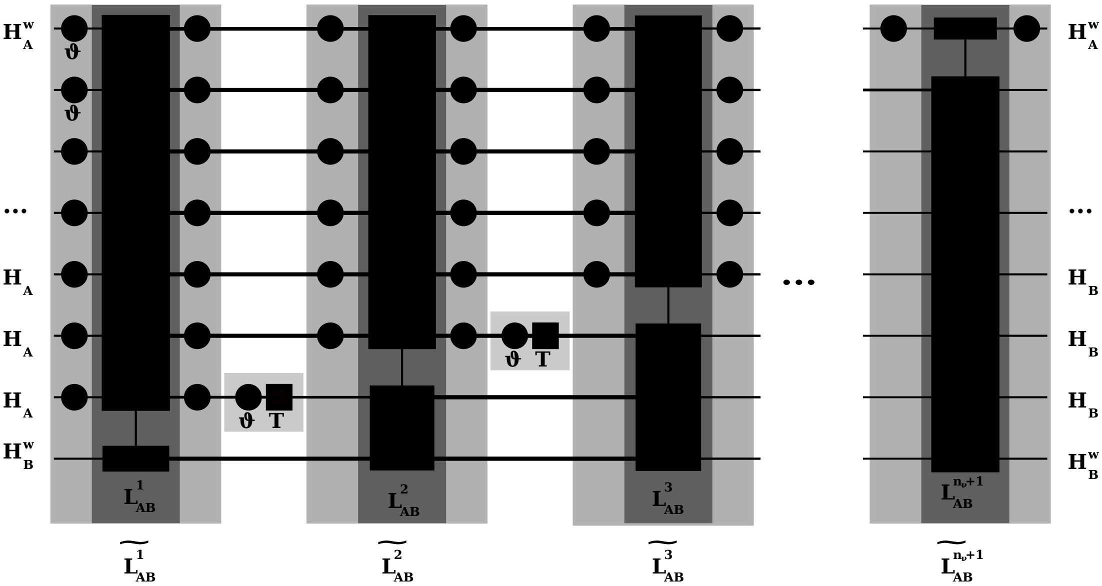

Figure 4.1.: Transpositions (black circles) are inserted in between the LOCC-maps $\mathcal{L}^k_{AB}$ and regrouped in the shown way. The lower part of the LOCC-maps corresponds to B and the upper part to A.

of rate $r \in \mathbb{R}^+$ for the two-way assisted quantum communication of the form

$$\mathcal{C}_{k^1_\nu,k^2_\nu}[\mathcal{T}] = \mathrm{tr}_W \circ \mathcal{L}^{k^1+1}_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_{k^1} \circ \mathcal{L}^{k^1}_{\mathrm{AB}} \circ \cdots \circ \widehat{\mathcal{T}}_2 \circ \mathcal{L}^2_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_1 \circ \mathcal{L}^1_{\mathrm{AB}} \circ \mathcal{E}_W \,,$$

according to Definition 12, such that $\Delta\left(k^1_\nu, k^2_\nu\right) = \inf \| \mathcal{C}_{k^1_\nu,k^2_\nu}[\mathcal{T}] - \mathrm{id}^{\otimes k^2_\nu} \|_\diamond \to 0$. Then we can do the following computation:

$$\begin{aligned} 2^{k^2_\nu} &= \|\vartheta^{\otimes k^2_\nu}\|_\diamond = \| \left(\mathrm{id}^{\otimes k^2_\nu} - \mathcal{C}_{k^1_\nu,k^2_\nu}[\mathcal{T}]\right) \circ \vartheta^{\otimes k^2_\nu} + \mathcal{T}_{k^1_\nu} \circ \vartheta^{\otimes k^2_\nu} \|_\diamond \\ &\leq \Delta\left(k^1_\nu, k^2_\nu\right) 2^{k^2_\nu} + \| \left(\mathrm{tr}_W \circ \mathcal{L}^{k^1_\nu+1}_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_{k^1_\nu} \circ \mathcal{L}^{k^1_\nu}_{\mathrm{AB}} \circ \cdots \circ \widehat{\mathcal{T}}_2 \circ \mathcal{L}^2_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_1 \circ \mathcal{L}^1_{\mathrm{AB}} \circ \mathcal{E}_W\right) \circ \vartheta^{\otimes k^2_\nu} \|_\diamond \,. \end{aligned}$$

Note that as we have $\mathcal{E}_W \circ \vartheta_A = (\mathrm{id}_B \otimes \vartheta_A) \circ \mathcal{E}_W$ and as $\mathcal{E}_W$ is a unital completely positive map and has therefore $\|\mathcal{E}_W\|_\diamond = 1$, we get

$$2^{k^2_\nu} \leq \Delta\left(k^1_\nu, k^2_\nu\right) 2^{k^2_\nu} + \|\mathrm{tr}_W \circ \mathcal{L}^{k^1_\nu+1}_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_{k^1_\nu} \circ \mathcal{L}^{k^1_\nu}_{\mathrm{AB}} \circ \cdots \circ \widehat{\mathcal{T}}_2 \circ \mathcal{L}^2_{\mathrm{AB}} \circ \widehat{\mathcal{T}}_1 \circ \mathcal{L}^1_{\mathrm{AB}} \circ (\mathrm{id}_B \otimes \vartheta_A) \|_\diamond \,.$$

Now we can insert transpositions in between the LOCC-maps, as sketched in Figure 4.1 and define new LOCC-maps $\tilde{\mathcal{L}}^k_{AB} = (\mathrm{id}_B \otimes \vartheta_A) \circ \mathcal{L}^k_{AB} \circ (\mathrm{id}_B \otimes \vartheta_A)$. Note that the number of Hilbert spaces corresponding to A and B changes in every step, as one of the spaces at A is transferred to B by a use of the channel $\mathcal{T}$, see Figure 4.1.

After regrouping we have

$$\begin{aligned}2^{k_\nu^2} &\leq \Delta\left(k_\nu^1, k_\nu^2\right) 2^{k_\nu^2} + \|\mathrm{tr}_W \circ \vartheta_W \circ \tilde{\mathcal{L}}_{\mathrm{AB}}^{k_\nu^1+1} \circ \widehat{\mathcal{T}\circ\vartheta}_{k_\nu^1} \circ \tilde{\mathcal{L}}_{\mathrm{AB}}^{k_\nu^1} \circ \cdots \circ \widehat{\mathcal{T}\circ\vartheta}_2 \circ \tilde{\mathcal{L}}_{\mathrm{AB}}^{2} \circ \widehat{\mathcal{T}\circ\theta}_1 \circ \tilde{\mathcal{L}}_{\mathrm{AB}}^{1}\|_\diamond \\ &\leq \Delta\left(k_\nu^1, k_\nu^2\right) 2^{k_\nu^2} + \|\left(\mathcal{T}\circ\vartheta\right)\|_\diamond^{k_\nu^1} ,\end{aligned}$$

where we used submultiplicativity of the cb-norm, see Appendix A, and that both $\tilde{\mathcal{L}}_{AB}^k$ and tr_W are unital completely positive maps and furthermore that $\mathrm{tr}_W \circ \theta_W = \mathrm{tr}_W$.

Now we can do the same argumentation, as in the original proof, and are done.

□

As a simple corollary we get:

Corollary 8. *For a quantum channel $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ we have*

$$\mathcal{D}_2\left(C_{\mathcal{T}}\right) \leq \mathcal{Q}_2\left(\mathcal{T}\right) \leq log\left(\|\vartheta \circ \mathcal{T}\|_\diamond\right) .$$

Proof. Trivial.

□

Thus we have recovered the standard distillation criterion for the Choi-matrix $C_{\mathcal{T}}$, because $C_{\mathcal{T}}$ is PPT iff $\vartheta \circ \mathcal{T}$ is a trace-preserving completely positive map iff $\log\left(\|\vartheta \circ \mathcal{T}\|_\diamond\right) = 0$.

The 2-way assisted quantum capacity and the distillability are strongly connected. A new bound on $\mathcal{Q}_2$ would also lead to a new distillation criterion for the states, which are Choi-matrices of quantum channels. In the next chapters we will state more results on how one might be able to construct such bounds.

4.2. Capacity Bounds Generated by Matrix Maps

In this section we will begin our study of capacity bounds, which are generated by a map $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_n$ in the same way, as the transposition bound is generated by the transposition.

By the results from the last section we see, that such a capacity bound will in some cases also lead to a distillability criterion in the sense of Definition 20. For distillability criteria generated by a map, see Definition 20, the assumption of multiplicativity or antimultiplicativity is to strong, because a hermitian preserving map has to be either completely positive or completely co-positive to have such a property. The goal of this section is to weaken these properties in order to generate a capacity bound or a distillation criterion.

In the proofs of Theorem 15 and Theorem 11 one can see, that multiplicativity or antimultiplicativity can be used, to move a map inside a certain completely positive map and leaving another completely positive map behind. More precisely we use that for every $\mathcal{R} \in \mathfrak{CPU}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$

there is a map $\mathcal{R}' \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$ such that

$$\mathcal{M} \circ \mathcal{R} = \mathcal{R}' \circ \mathcal{M} .$$

In the case of the transposition the Kraus operators of $\mathcal{R}'$ are the transposed ones of $\mathcal{R}$, as shown in Lemma 6. Note that we switched to Heisenberg picture here, using unital completely positive maps.

This property has been studied by Smith, et al. [GS12] and they show, that there are only two maps up to unitary conjugation fulfilling this property.

The central point in their argument is the following theorem, which we will prove using Theorem 7 about unitary equivalence preservers.

Theorem 16. *Let $\mathcal{M} \in \mathfrak{HP}(\mathfrak{M}_n, \mathfrak{M}_n)$ be an invertible map with $A := \mathcal{M}(I_n) > 0$ fulfilling*

$$\mathcal{M} \circ \mathcal{K}_U \circ \mathcal{M}^{-1} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$$

for all unitaries U, where $\mathcal{K}_U(X) = UXU^\dagger$. Then there is $a \in \mathbb{R}$ and a unitary U that either

$$\begin{aligned} \mathcal{M}(X) &= aA^{1/2}UXU^\dagger A^{1/2} + \frac{1-a}{n} tr(X) A \\ &\textit{or} \\ \mathcal{M}(X) &= aA^{1/2}UX^TU^\dagger A^{1/2} + \frac{1-a}{n} tr(X) A . \end{aligned}$$

Proof. Assume first that $A := \mathcal{M}(I) = I$. For an arbitrary unitary U we have by the assumption that $\mathcal{M} \circ \mathcal{K}_U \circ \mathcal{M}^{-1}$ is a unital completely positive map. Now we can look at the inverse of the above maps and we obtain

$$\left(\mathcal{M} \circ \mathcal{K}_U \circ \mathcal{M}^{-1}\right)^{-1} = \mathcal{M} \circ \mathcal{K}_U^{-1} \circ \mathcal{M}^{-1} = \mathcal{M} \circ \mathcal{K}_{U^\dagger} \circ \mathcal{M}^{-1} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n) .$$

As the inverse is also a positive map, we can apply Corollary 5, and obtain a unitary $V \in \mathfrak{U}_n$, such that

$$\mathcal{M} \circ \mathcal{K}_U \circ \mathcal{M}^{-1} = \mathcal{K}_V .$$

Note that the case corresponding to the transposition does not matter, as $\mathcal{M} \circ \mathcal{K}_U \circ \mathcal{M}^{-1} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$ by assumption. Finally we obtain

$$\mathcal{M} \circ \mathcal{K}_U = \mathcal{K}_V \circ \mathcal{M} ,$$

which shows that $\mathcal{M}$ preserves unitary equivalence. By first restricting $\mathcal{M}$ to hermitian matrices and then using Lemma 4 and Theorem 7 we get the existence of $a, b \in \mathbb{R}$ and U unitary such that either

$$\mathcal{M}|_{\mathfrak{H}_n}(X) = aUXU^\dagger + bI\mathrm{tr}(X)$$
or
$$\mathcal{M}|_{\mathfrak{H}_n}(X) = aUX^TU^\dagger + bI\mathrm{tr}(X)\ .$$

Using the Cartesian decomposition argument from the proof of Theorem 5 this shows, that also the unrestricted $\mathcal{M}$ is of the above form. As $\mathcal{M}$ has to be unital by assumption, we obtain the condition $b = \frac{1-a}{n}$. Using Remark 9 shows $\mathcal{M}$ to be bijective, which finishes the proof for $\mathcal{M}(I) = I$.

From $A := \mathcal{M}(I) > 0$ we obtain $A^{-1} > 0$ using spectral decomposition. With this we can define the unital map

$$\tilde{\mathcal{M}}(X) := A^{-1/2}\mathcal{M}(X)A^{-1/2}\ ,$$

which has all the properties from the assumptions. From the above proof for unital maps, we know that $\tilde{\mathcal{M}}$ is a unitary equivalence preserver. This shows, that there is $a \in \mathbb{R}$ and a unitary matrix U such that either

$$\mathcal{M}(X) = aA^{1/2}UXU^\dagger A^{1/2} + \frac{1-a}{n}\mathrm{tr}(X)A$$
or
$$\mathcal{M}(X) = aA^{1/2}UX^TU^\dagger A^{1/2} + \frac{1-a}{n}\mathrm{tr}(X)A\ ,$$

which finishes the general case.

□

The property studied by Smith et al. [GS12] leads exactly to

$$\mathcal{M} \circ \mathcal{R} \circ \mathcal{M}^{-1} = \mathcal{R}' \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$$

for all $\mathcal{R} \in \mathfrak{CPU}(\mathfrak{M}_n, \mathfrak{M}_n)$, which gives their result by using the theorem.

Note that this result is similar to the results from chapter 3. But that the property used in the above theorem is more general than multiplicativity or antimultiplicativity used there.

Maps fulfilling the above property for unitary conjugations have to be already of a particular form. This is however only a necessary condition and we have not showed, that the above maps fulfill the stronger property for all unital completely positive maps. Thus we will do some explicit calculations for the first case, as the second case involving the transposition works in analogy.

It is easy to see, that the inverse of the map $\mathcal{M}(X) = aA^{1/2}UXU^\dagger A^{1/2} + \frac{1-a}{n}\mathrm{tr}(X)A$ is given by

$$\mathcal{M}^{-1}(Y) = \frac{1}{a}U^\dagger A^{-1/2}YA^{-1/2}U - \frac{1-a}{na}\mathrm{tr}\left(A^{-1/2}XA^{-1/2}\right)I_n\,,$$

where we used, that $A^{1/2}$ is invertible and hermitian. From this we can compute

$$\begin{aligned}\mathcal{M}\circ\mathcal{R}\circ\mathcal{M}^{-1}(Y) = {} & A^{1/2}U\mathcal{R}\left(U^\dagger A^{-1/2}YA^{-1/2}U\right)U^\dagger A^{1/2}\\ & + \frac{1-a}{na}\mathrm{tr}\left(\mathcal{R}\left(U^\dagger A^{-1/2}YA^{-1/2}U\right)\right)A\\ & - \frac{1-a}{n}\mathrm{tr}\left(A^{-1/2}YA^{-1/2}\right)A^{1/2}U\mathcal{R}(I_n)U^\dagger A^{1/2}\\ & - \frac{(1-a)^2}{an^2}\mathrm{tr}(\mathcal{R}(I_n))\,\mathrm{tr}\left(A^{-1/2}YA^{-1/2}\right)A\,,\end{aligned}$$

for $\mathcal{R}\in\mathfrak{CP}(M_n,M_n)$.

Consider the map $\mathcal{R}\in\mathfrak{CPU}(M_n,M_n)$ defined by

$$\mathcal{R}(X) = \langle u|X|u\rangle I_n\,,$$

for some normalized vector $|u\rangle\in\mathbb{C}^n$. Now we choose $Y = A^{1/2}U|v\rangle\langle v|U^\dagger A^{1/2}\geq 0$ for some normalized $|v\rangle\in\mathbb{C}^n$ and such that $\langle u|v\rangle = 0$. With this, we have $\mathcal{R}\left(U^\dagger A^{-1/2}YA^{-1/2}U\right) = 0$, which cancels the two first terms in the above expression and we obtain

$$\mathcal{M}\circ\mathcal{R}\circ\mathcal{M}^{-1}(Y) = -\frac{1-a}{an}A^{1/2}A^{1/2}\,.$$

As this expression is negative whenever $\frac{1-a}{a}>0$, the map $\mathcal{M}\circ\mathcal{R}\circ\mathcal{M}^{-1}$ cannot be completely positive when $0<a<1$.

Next we want to assume the above property for completely positive trace-preserving maps, instead of unital maps. As the unitary conjugations are both unital and trace-preserving, we can apply the above theorem for calculations in Heisenberg and in Schrödinger picture.

Consider the map $\mathcal{R}\in\mathfrak{CPTP}(M_n,M_n)$ defined by

$$\mathcal{R}(X) = \mathrm{tr}(X)|u\rangle\langle u|\,,$$

for some normalized vector $|u\rangle\in\mathbb{C}^n$. By inserting this in the general formula above, we obtain

$$\begin{aligned}\mathcal{M}\circ\mathcal{R}\circ\mathcal{M}^{-1}(Y) = {} & a\,\mathrm{tr}\left(A^{-1/2}YA^{-1/2}\right)A^{1/2}U|u\rangle\langle u|U^\dagger A^{1/2}\\ & + \frac{1-a}{n}\mathrm{tr}\left(A^{-1/2}YA^{-1/2}\right)A\,.\end{aligned}$$

The first term in the above expression is of rank 1 and the last term of rank n, because $A\in\mathfrak{M}_n$ was assumed to be invertible. Thus the above expression can only be positive if $a\leq 1$.

Furthermore we need $a + \frac{1-a}{n} \geq 0$ for the above matrix to be positive. This is fulfilled iff $a \geq -\frac{1}{n-1}$.

By the above calculations we showed, that if the maps from our theorem fulfill the property

$$\mathcal{M} \circ \mathcal{R} \circ \mathcal{M}^{-1} = \mathcal{R}' \in \mathfrak{CP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$$

for all $R \in \mathfrak{CPU}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$, then we must have $a \geq 1$ or $a < 0$ and if it is fulfilled for all $R \in \mathfrak{CPTP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$, then we must have $a \in \left[-\frac{1}{n-1}, 1\right]$. In particular if it is fulfilled for all $R \in \mathfrak{CP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$, we obtain $a = 1$ as the only positive value, which corresponds to either

$$\mathcal{M}\left(X\right) = A^{1/2} U X U^{\dagger} A^{1/2}$$
$$\text{or}$$
$$\mathcal{M}\left(X\right) = A^{1/2} U X^{T} U^{\dagger} A^{1/2} .$$

It is clear, that those maps fulfill the property. We conjecture that this are the only possibilities, where the property is fulfilled for allowing unital and trace-preserving completely positive maps.

We can prove, using a simple argument, that this is true for the case of arbitrary completely positive maps, i.e.

$$\mathcal{M} \circ \mathcal{R} \circ \mathcal{M}^{-1} = \mathcal{R}' \in \mathfrak{CP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$$

for all $R \in \mathfrak{CP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$. This property is equivalent to

$$\left(\vartheta_n \circ \left(\mathcal{M}^*\right)^{-1} \circ \vartheta_n \otimes \mathcal{M}\right) \geq 0$$

by using the Choi-Jamiolkowski-Isomorphism. We also used the fact that every positive matrix on a tensor product can be written as the Choi-matrix of a completely positive map and Lemma 5. The above is fulfilled iff $\mathcal{M}$ and $\mathcal{M}^{-1}$ is positive, which leaves using Corollary 5 only the two possibilities stated above. This can be seen as a generalization of the results from chapter 3.

We can interpret this result in another way, connecting it to the theory of mcs-cones, which we introduced in chapter 2. For a linear map $\mathcal{M} : \mathfrak{M}_n \rightarrow \mathfrak{M}_n$ consider the cone

$$\mathfrak{C}_{\mathcal{M}} := \left\{\mathcal{M} \circ \mathcal{R} : \mathcal{R} \in \mathfrak{CP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)\right\}\right\} .$$

This cone is a direct generalization of the mcs-cone of completely co-positive maps, but taking the map $\mathcal{M}$ instead of the transposition. A natural question is, for which maps $\mathcal{M}$ such a cone is also an mcs-cone.

Note that $\mathcal{M} \in \mathfrak{C}_{\mathcal{M}}$ as $\text{id}_n \in \mathfrak{CP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$. Therefore $\mathcal{M}$ has to be positive, if we want $\mathfrak{C}_{\mathcal{M}}$ to be an mcs-cone, as those contain only positive elements. In this case $\mathcal{M}$ is also hermiticity preserving. The mcs-property from chapter 2, applied to the cone $\mathfrak{C}_{\mathcal{M}}$, states that for every

$\mathcal{R} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$ there is an $\mathcal{S} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$ such that

$$\mathcal{M} \circ \mathcal{R} = \mathcal{S} \circ \mathcal{M} \,,$$

which is exactly the above property. Therefore we get the following corollary:

Corollary 9. *Let $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_n$ be a linear, invertible map, with $A := \mathcal{M}(I_n) > 0$, such that $\mathfrak{C}_{\mathcal{M}}$ is an mcs-cone. Then there is a unitary matrix U such that either*

$$\mathcal{M}(X) = A^{1/2} U X U^\dagger A^{1/2}$$
$$\text{or}$$
$$\mathcal{M}(X) = A^{1/2} U X^T U^\dagger A^{1/2} \,.$$

This shows, that our earlier attempt of generalizing the mcs-cone of completely co-positive maps did not lead to any new mcs-cones. It is easy to see, that the first of the above maps leads to the cone of completely positive maps and the second one to the cone of completely co-positive maps. Thus the transposition and the identity are the only maps for which this construction yields mcs-cones.

4.3. Tensor-stable Positive Maps

After studying general properties of maps generating capacity bounds we will try to construct concrete examples of such bounds. A property that will be essential in our construction is the following.

Definition 22. (*Tensor-stable positive*)
*For $k \in \mathbb{N}$ we call a map $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_m$ **k-tensor-stable positive** if $M^{\otimes k}$ is positive. If M is k-tensor-stable positive for all $k \in \mathbb{N}$, we call it **tensor-stable positive**.*

Note that every map in $\mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_m)$ or $\mathfrak{coCP}(\mathfrak{M}_n, \mathfrak{M}_m)$ is tensor-stable positive. It is also easy to see, that for $k, k' \in \mathbb{N}$ with $k < k'$ every k'-tensor-stable positive map is also k-tensor-stable positive.

In this chapter we will assume that there is a tensor-stable positive map $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_m$, which is neither in $\mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_m)$ nor in $\mathfrak{coCP}(\mathfrak{M}_n, \mathfrak{M}_m)$. We do not know at the time of writing, whether such a map exists and in the final section we will discuss some possibilities of constructing such maps. In particular we will construct an example of a qubit-map $\mathcal{M} : \mathfrak{M}_2 \to \mathfrak{M}_2$ that is 2-tensor-stable positive but neither in $\mathfrak{CP}(\mathfrak{M}_2, \mathfrak{M}_2)$ nor in $\mathfrak{coCP}(\mathfrak{M}_2, \mathfrak{M}_2)$.

4.3.1. Capacity Bounds via Tensor-stable Positive Maps

Before we state our main theorem we will prove the following lemma which will allow us, to obtain unital tensor-stable maps from milder assumptions.

Lemma 10. *For $k, l \in \mathbb{N}$ let $\mathcal{M} \in \mathfrak{P}_l(\mathfrak{M}_n, \mathfrak{M}_m) \setminus \mathfrak{P}_{l+1}(\mathfrak{M}_n, \mathfrak{M}_m)$ be k-tensor-stable positive. Then there is a map $\widehat{\mathcal{M}} \in \mathfrak{P}_l(\mathfrak{M}_n, \mathfrak{M}_m) \setminus \mathfrak{P}_{l+1}(\mathfrak{M}_n, \mathfrak{M}_m)$ which is k-tensor-stable positive and unital.*

Proof. We define $r := \operatorname{rank}(\mathcal{M}(I_n))$. As $\mathcal{M} \in \mathfrak{P}_l(\mathfrak{M}_n, \mathfrak{M}_m)$ we have $\mathcal{M}(I_n) \geq 0$. From this we get $\mathcal{M}(I_n)^{-1} \geq 0$ where we computed the Moore-Penrose pseudoinverse of the matrix $\mathcal{M}(I_n)$. Thus we can define the map $\widehat{\mathcal{M}} \in \mathfrak{P}_l(\mathfrak{M}_n, \mathfrak{M}_m)$ via

$$\widehat{\mathcal{M}}(X) = \mathcal{M}(I_n)^{-1/2} \mathcal{M}(X) \mathcal{M}(I_n)^{-1/2} \oplus \frac{\operatorname{tr}(X)}{m} I_{m-r} .$$

Note that we have

$$\mathcal{M}(I_n)^{-1} \mathcal{M}(I_n) = I_r \oplus 0_{m-r} ,$$

which shows that $\widehat{\mathcal{M}}$ defined in the above way is indeed unital. It is easy to see, that $\widehat{\mathcal{M}}$ also has the desired positivity properties, as $\widehat{\mathcal{M}} = \left(\mathcal{R}_{\mathcal{M}(I)^{-1/2}} \circ \mathcal{M}\right) \oplus \mathcal{P}$ for the completely positive maps $\mathcal{R}_{\mathcal{M}(I)^{-1/2}}(X) = \mathcal{M}(I_n)^{-1/2} X \mathcal{M}(I_n)^{-1/2}$ and $\mathcal{P}(X) = \frac{\operatorname{tr}(X)}{m} I_{m-r}$.

□

Now we are in the position to prove our main theorem about capacity bounds generated by tensor-stable positive maps. Note that we are in Heisenberg picture here, meaning that a quantum channel is a unital completely positive map.

Theorem 17. *Let $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_n$ be an invertible, trace-preserving, tensor-stable positive map, which is not completely positive. Then we get the following bound on the quantum capacity*

$$\mathcal{Q}(\mathcal{T}) \leq \frac{log\left(\|\mathcal{T} \circ \mathcal{M}^{-1}\|_{cb} \|\mathcal{M}(I)\|_{\infty}\right) log(n)}{log\left(\|\vartheta \circ \mathcal{M}^* \circ \vartheta\|_{cb}\right)}$$

for every quantum channel $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_n$.

Proof. We will choose an n-dimensional identity as a reference for the quantum capacity. By [DK04] this corresponds to a change of units and to the factor of $\log(n)$ at the right hand side of our bound.

Take an achievable rate $r \in \mathbb{R}^+$ for the channel $\mathcal{T}$ and sequences $\left(k_\nu^1\right)_{\nu \in \mathbb{N}}$, $\left(k_\nu^2\right)_{\nu \in \mathbb{N}}$ such that $\frac{k_\nu^1}{k_\nu^2} \to r$ for $\nu \to \infty$. Denote by $\mathcal{E}_\nu, \mathcal{D}_\nu$ suitable encoding and decoding maps, i.e. quantum channels such that $\Delta\left(k_\nu^1, k_\nu^2\right) = \|\mathrm{id}_d^{\otimes k_\nu^1} - \mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\|_{\mathrm{cb}} \to 0$ for $\nu \to \infty$.

We set $a := \|\vartheta \circ \mathcal{M}^* \circ \vartheta\|_{\mathrm{cb}} > 1$, because $\mathcal{M}$ is not completely positive, but trace-preserving, see Appendix A.

$$
\begin{aligned}
a^{k_\nu^1} &= \| \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1} \|_{\text{cb}} \\
&= \| \left(\text{id}_d^{\otimes k_\nu^1} - \mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\right) \circ \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1} + \left(\mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\right) \circ \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1} \|_{\text{cb}} \\
&\leq \| \left(\text{id}_d^{\otimes k_\nu^1} - \mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\right) \circ \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1} \|_{\text{cb}} + \| \left(\mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu\right) \circ \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1} \|_{\text{cb}} \\
&\leq \Delta\left(k_\nu^1, k_\nu^2\right) a^{k_\nu^1} + \|\mathcal{D}_\nu \circ \mathcal{T}^{\otimes k_\nu^2} \circ \left(\mathcal{M}^{-1}\right)^{\otimes k_\nu^2} \circ \mathcal{M}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu \circ \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1} \|_{\text{cb}} \\
&\leq \Delta\left(k_\nu^1, k_\nu^2\right) a^{k_\nu^1} + \| \left(\mathcal{T} \circ \mathcal{M}^{-1}\right)^{\otimes k_\nu^2} \|_{\text{cb}} \|\mathcal{M}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu \circ \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1} \|_{\text{cb}} \\
&= \Delta\left(k_\nu^1, k_\nu^2\right) a^{k_\nu^1} + \|\mathcal{T} \circ \mathcal{M}^{-1}\|_{\text{cb}}^{k_\nu^2} \|\mathcal{M}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu \circ \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1} \|_{\text{cb}} \,,
\end{aligned}
$$

where we used submultiplicativity of the cb-norm, the fact that $\|\mathcal{D}\|_{\text{cb}} = 1$ for a quantum channel $\mathcal{D}$ and the cross-norm property of the cb-norm. From the tensor-stable positivity we obtain that $\mathcal{M}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu \circ (\vartheta \circ \mathcal{M}^* \circ \vartheta)$ is a completely positive map for every quantum channel $\mathcal{E}_\nu$. In particular we have

$$
\left(\text{id}_{k_\nu^1} \otimes \mathcal{M}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu \circ \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1}\right)(\omega) = \mathcal{M}^{\otimes k_\nu^1 k_\nu^2}\left(C_{\mathcal{E}_\nu}\right) \geq 0 \,,
$$

because $\mathcal{M}$ is tensor-stable positive and $\mathcal{E}$ is completely positive. This allows us to replace the cb-norm by the $\infty-$norm at I.

By using unitality of both $\mathcal{M}^*$ and $\mathcal{E}$ we get

$$
\begin{aligned}
& a^{k_\nu^1}\left(1 - \Delta\left(k_\nu^1, k_\nu^2\right)\right) \leq \|\mathcal{T} \circ \mathcal{M}^{-1}\|_{\text{cb}}^{k_\nu^2} \|\mathcal{M}^{\otimes k_\nu^2} \circ \mathcal{E}_\nu \circ \left(\vartheta \circ \mathcal{M}^* \circ \vartheta\right)^{\otimes k_\nu^1}(I)\|_\infty = \left(\|\mathcal{T} \circ \mathcal{M}^{-1}\|_{\text{cb}} \|\mathcal{M}(I)\|_\infty\right)^{k_\nu^2} \\
\implies & \frac{k_\nu^1}{k_\nu^2} + \frac{\log\left(1 - \Delta\left(k_\nu^1, k_\nu^2\right)\right)}{\log(a)\, k_\nu^2} \leq \frac{\log\left(\|\mathcal{T} \circ \mathcal{M}^{-1}\|_{\text{cb}} \|\mathcal{M}(I)\|_\infty\right)}{\log(a)}
\end{aligned}
$$

after taking the logarithm. In the limit $\nu \to \infty$ this leads to

$$
r \leq \frac{\log\left(\|\mathcal{T} \circ \mathcal{M}^{-1}\|_{\text{cb}} \|\mathcal{M}(I)\|_\infty\right)}{\log(a)} \,,
$$

which gives the desired result as this holds for all achievable rates $r \in \mathbb{R}^+$.

□

Note that the existence of an invertible, tensor-stable positive map $\mathcal{M} : \mathfrak{M}_n \to \mathfrak{M}_n$ would also give rise to a quantum capacity bound. From such a map we can construct a unital, invertible, tensor-stable positive map $\widehat{\mathcal{M}} : \mathfrak{M}_n \to \mathfrak{M}_n$ with the same positivity properties. This is done by applying Lemma 10 to $\mathcal{M}^*$ and by noting, that taking the adjoint does not influence the positivity of the map.

We will now try to estimate the above bound by simpler expressions.

An application of Theorem 24 in Appendix A and positivity of the occurring maps yields

$$\begin{aligned}\|\vartheta \circ \mathcal{M}^* \circ \vartheta\|_{\mathrm{cb}} &\leq n\|\vartheta \circ \mathcal{M}^* \circ \vartheta\|_{\infty\to\infty} \\ &= n\|\vartheta \circ \mathcal{M}^*\left(I_n\right)\|_{\infty} \\ &= n\,,\end{aligned}$$

where we used that $\mathcal{M}$ is trace-preserving.

This leads to the simple estimate

$$\frac{\log\left(\|\mathcal{T}\circ\mathcal{M}^{-1}\|_{\mathrm{cb}}\|\mathcal{M}\left(I\right)\|_{\infty}\right)\log\left(n\right)}{\log\left(\|\vartheta\circ\mathcal{M}^*\circ\vartheta\|_{\mathrm{cb}}\right)} \geq \log\left(\|\mathcal{T}\circ\mathcal{M}^{-1}\|_{\mathrm{cb}}\|\mathcal{M}\left(I\right)\|_{\infty}\right)\,.$$

Consider now the case of an invertible $\mathcal{M} \in \mathbf{co}\mathfrak{CP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$, i.e. $\mathcal{M} = \mathcal{R}\circ\vartheta_n$ for some invertible $\mathcal{R} \in \mathfrak{CPTP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$. A completely co-positive map $\mathcal{M}$ is of course also tensor-stable positive. By Theorem 23 from Appendix A, we obtain

$$\|\mathcal{M}\left(I_n\right)\|_{\infty} = \|\mathcal{R}\left(I_n\right)\|_{\infty} = \|\mathcal{R}\|_{\mathrm{cb}}\,.$$

Therefore by using submultiplicativity of the cb-norm and monotonicity of the logarithm, we get from the estimate we derived above:

$$\begin{aligned}\frac{\log\left(\|\mathcal{T}\circ\mathcal{M}^{-1}\|_{\mathrm{cb}}\|\mathcal{M}\left(I\right)\|_{\infty}\right)\log\left(n\right)}{\log\left(\|\vartheta\circ\mathcal{M}^*\circ\vartheta\|_{\mathrm{cb}}\right)} &\geq \log\left(\|\mathcal{T}\circ\mathcal{M}^{-1}\|_{\mathrm{cb}}\|\mathcal{M}\left(I\right)\|_{\infty}\right) \\ &= \log\left(\|\mathcal{T}\circ\vartheta_n\circ\mathcal{R}^{-1}\|_{\mathrm{cb}}\|\mathcal{R}\|_{\mathrm{cb}}\right) \\ &\geq \log\left(\|\mathcal{T}\circ\vartheta_n\circ\mathcal{R}^{-1}\circ\mathcal{R}\|_{\mathrm{cb}}\right) \\ &= \log\left(\|\mathcal{T}\circ\vartheta_n\|_{\mathrm{cb}}\right)\,.\end{aligned}$$

This shows that for an invertible $\mathcal{M} \in \mathbf{co}\mathfrak{CP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$ our bound is worse than the known transposition bound and therefore not interesting. Note that if we had started with an arbitrary not necessarily trace-preserving invertible $\mathcal{M} \in \mathbf{co}\mathfrak{CP}\left(\mathfrak{M}_n, \mathfrak{M}_n\right)$ and constructed a bound using Lemma 10 this would have also lead to a bound worse than the transposition bound. This is clear as we would have to construct a trace-preserving map to do the proof.

Now we consider the case, where our bound is better than the transposition bound for all quantum channels. If this is the case for a unital $\mathcal{M}$, then $\mathcal{M}\left(X\right) = UX^TU^\dagger$ for a unitary $U \in \mathfrak{U}_n$ and therefore the bound coincides with the ordinary transposition bound. This shows that if we want to find a new bound using a unital and trace-preserving tensor-stable positive map, the bound has to be worse than the transposition at least in some cases.

To show this it suffices to assume, that our bound detects all entanglement breaking channels to have zero quantum capacity. This means, that

$$\|\mathcal{T}\circ\mathcal{M}^{-1}\|_{\mathrm{cb}}\|\mathcal{M}\left(I\right)\|_{\infty} = 1$$

for all entanglement breaking channels $\mathcal{T} \in \mathfrak{CPU}(\mathfrak{M}_n, \mathfrak{M}_n)$, note that we are in Heisenberg picture here. If $\mathcal{M}$ is unital, we have

$$\|\mathcal{T} \circ \mathcal{M}^{-1}\|_{\mathrm{cb}} = 1$$

for the unital linear maps $\mathcal{T} \circ \mathcal{M}^{-1}$ for all entanglement breaking channels $\mathcal{T}$. Theorem 21 shows that we have $\mathcal{T} \circ \mathcal{M}^{-1} \in \mathfrak{CPU}(\mathfrak{M}_n, \mathfrak{M}_n)$ for all entanglement breaking channels $\mathcal{T}$. We will now use the characterization Theorem 10 for mcs-cones to show that $\mathcal{M}^{-1}$ has to be positive. Corollary 5 and using that $\mathcal{M}$ is not completely positive would then lead to the existence of an invertible $S \in \mathfrak{M}_n$ such that

$$\mathcal{M}(X) = SX^T S^\dagger \,,$$

which finishes the proof. Note that S has to be unitary by unitality of $\mathcal{M}$. In order to apply Theorem 10, we have to show, that

$$\mathcal{T} \circ \mathcal{M}^{-1} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$$

for all $\mathcal{T} \in \mathfrak{SP}_1(\mathfrak{M}_n, \mathfrak{M}_n)$. We proved this already for all unital $\mathcal{T} \in \mathfrak{SP}_1(\mathfrak{M}_n, \mathfrak{M}_n)$, i.e. entanglement breaking channels. For an arbitrary $\mathcal{T} \in \mathfrak{SP}_1(\mathfrak{M}_n, \mathfrak{M}_n)$ consider the unital map $\widehat{\mathcal{T}} : \mathfrak{M}_n \to \mathfrak{M}_n$, defined by

$$\widehat{\mathcal{T}}(X) = \mathcal{T}(I_n)^{-1/2}\, \mathcal{T}(X)\, \mathcal{T}(I_n)^{-1/2} \,,$$

which is an entanglement breaking quantum channel. By using the above calculations for entanglement breaking channels and the fact, that $\mathcal{R}(X) = \mathcal{T}(I_n)^{1/2} X \mathcal{T}(I_n)^{1/2}$ defines a completely positive map, we see that $\mathcal{T} \circ \mathcal{M}^{-1} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$ for all $\mathcal{T} \in \mathfrak{SP}_1(\mathfrak{M}_n, \mathfrak{M}_n)$, which finishes the proof by the argument stated above.

We can obtain a bound on the quantum capacity from a tensor-stable positive map. It is however clear from the above discussion, that using trace-preserving completely co-positive maps leads to bounds worse than the transposition bound. Therefore we have to use non-trivial tensor-stable positive maps, i.e. maps which are neither completely positive nor completely co-positive. We do not know, whether such maps exist. In the next section, we show two ways, which could yield methods for constructing such maps. Furthermore we give examples for nontrivial 2-tensor-stable positive maps, which shows, that the question of existence cannot be decided on this level.

4.4. Examples for Nontrivial 2-Tensor-stable Positivity

In this chapter we will explain how to construct non-trivial, i.e. neither completely positive nor completely co-positive, 2-tensor-stable positive maps. This shows, that there is a possibility of constructing new quantum capacity bounds by the construction stated in the last section.

4.4.1. Dual Pair Construction

We will start with a slightly different problem, namely to find examples of pairs $(\mathcal{R}, \mathcal{S})$ of maps $\mathcal{R}, \mathcal{S} : \mathfrak{M}_n \to \mathfrak{M}_n$ such that $\mathcal{R} \otimes \mathcal{S} : \mathfrak{M}_n \otimes \mathfrak{M}_n \to \mathfrak{M}_n \otimes \mathfrak{M}_n$ is a positive map.

Definition 23. (*Dual Pair*)
*We call a pair $(\mathcal{R}, \mathcal{S})$ of maps $\mathcal{R}, \mathcal{S} : \mathfrak{M}_n \to \mathfrak{M}_n$ a **dual pair** if there is a symmetric mcs-cone $\mathfrak{C}$ such that $\mathcal{R} \in \mathfrak{C}$ and $\mathcal{S} \in \mathfrak{C}^\circ$.*

From this we get the following theorem

Theorem 18. *For a dual pair $(\mathcal{R}, \mathcal{S})$ of maps $\mathcal{R}, \mathcal{S} : \mathfrak{M}_n \to \mathfrak{M}_n$ the map $\mathcal{R} \otimes \mathcal{S}$ is positive.*

Proof. It suffices to check positivity for pure states. Every pure state $|\psi\rangle \langle\psi| \in \mathfrak{M}_n \otimes \mathfrak{M}_n$ can be written in the form $|\psi\rangle \langle\psi| = (\mathrm{id} \otimes \mathcal{C})(\omega_n)$ for a maximally entangled state ω_n and a completely positive map $\mathcal{C}$. From this we get

$$\begin{aligned}(\mathcal{R} \otimes \mathcal{S})(|\psi\rangle \langle\psi|) &= (\mathcal{R} \otimes \mathcal{S} \circ \mathcal{C})(\omega) \\ &= (\mathrm{id} \otimes (\mathcal{S} \circ \mathcal{C} \circ \vartheta \mathcal{R}^* \vartheta))(\omega) \\ &= \left(\mathrm{id} \otimes \mathcal{S} \circ \widehat{\mathcal{R}}\right)(\omega) \geq 0\,.\end{aligned}$$

Here we used Lemma 5. We also used, that the considered mcs-cones are symmetric, which means that $\vartheta \circ \mathcal{R}^* \circ \vartheta \in \mathfrak{C}$ whenever $\mathcal{R} \in \mathfrak{C}$, and that for every completely positive map $\mathcal{C}$, $\mathcal{C} \circ \mathcal{R} \in \mathfrak{C}$ whenever $\mathcal{R} \in \mathfrak{C}$. In the last step we used Theorem 10, which gives that $\mathcal{S} \circ \widehat{\mathcal{R}}$ is completely positive.

□

A trivial example for this would be to choose the two maps to be completely positive. As $\mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)^\circ = \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$ this fulfills our assumptions.

Another less trivial example is to choose $\mathfrak{C} = \mathcal{P}_k(\mathfrak{M}_n, \mathfrak{M}_n)$ the cone of k-positive maps, or more general a typical mcs-cone and its dual. This shows that for every pair $(\mathcal{R}, \mathcal{S})$, where $\mathcal{R}$ is k-positive and $\mathcal{S}$ is k-superpositive, $\mathcal{R} \otimes \mathcal{S}$ is positive. Of course we still have $\mathcal{S} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n)$.

In general this construction always yields one of the maps to be completely positive or completely co-positive, when applied to the typical mcs-cones. But in the next section we will see an example of a 2-tensor-stable positive map which is neither completely positive nor completely co-positive.

4.4.2. Tensor-stable Positivity from Entanglement Annihilating Maps

In this section we will use a different approach to construct tensor-stable positive maps, using entanglement annihilating channels, which have been introduced by Ziman et al. [LM10].

Definition 24. (*Entanglement Annihilating maps, see [LM10]*)
A quantum channel $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_{m_1} \otimes \cdots \otimes M_{m_k}$ is called ***entanglement annihilating*** *if $\mathcal{T}(X)$ is separable for all quantum states $X \in \mathfrak{M}_n$.*
If the channel $\mathcal{T}^{\otimes n}$ is entanglement annihilating, the local channel $\mathcal{T}$ is called ***n-locally entanglement annihilating*** *(**n-LEA**).*

The characterization of all entanglement annihilating or even the n-LEA is not known for general dimensions, but in the case of qubit channels and 2-LEA several results have been proven by Filippov et al. [SF12] by doing explicit calculations. Two of them, which are useful in our case, are the following.

Proposition 1. ([*SF12*])
A unital qubit channel $\mathcal{T}$ is 2-LEA iff $\mathcal{T}^2$ is entanglement breaking.

This proposition characterizes the unital qubit 2-LEA quantum channels completely. It also shows, that there are 2-LEA qubit channels which are not entanglement breaking, as the condition, that $\mathcal{T}^2$ is entanglement-breaking, does not imply $\mathcal{T}$ to be entanglement breaking itself.

When we connect two 2-LEA channels together using a tensor product, one can show the following important result:

Proposition 2. ([*SF12*])
For unital qubit 2-LEA channels $\mathcal{T}_1, \mathcal{T}_2$ we have that $\mathcal{T}_1 \otimes \mathcal{T}_2$ is entanglement annihilating.

This shows that 2-LEA channels can be considered as building blocks for entanglement annihilating channels, and we will use this fact in our construction.

In order to construct a 2-tensor-stable positive qubit map consider the decomposable, and thus positive map

$$\mathcal{T} = \mathcal{T}_1 + \vartheta \circ \mathcal{T}_2$$

for unital 2-LEA qubit channels $\mathcal{T}_1$ and $\mathcal{T}_2$. One immediately sees that

$$\mathcal{T} \otimes \mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_1 + (\mathrm{id} \otimes \vartheta) \circ (\mathcal{T}_1 \otimes \mathcal{T}_2) + (\vartheta \otimes \mathrm{id}) \circ (\mathcal{T}_2 \otimes \mathcal{T}_1) + \vartheta \circ (\mathcal{T}_2 \otimes \mathcal{T}_2) \ .$$

Because we used 2-LEA channels as building blocks we can use Proposition 2, and conclude that $\mathcal{T}_i \otimes \mathcal{T}_j$ is entanglement annihilating, and therefore $\mathcal{T}_i \otimes \mathcal{T}_j (X)$ is separable for $i, j \in \{1, 2\}$. As a separable state is also PPT this means that $\mathcal{T} \otimes \mathcal{T}$ is a sum of positive maps and therefore positive too.

One has to choose $\mathcal{T}_1, \mathcal{T}_2 \notin \mathfrak{CP}(\mathfrak{M}_2, \mathfrak{M}_2) \cap \mathfrak{coCP}(\mathfrak{M}_2, \mathfrak{M}_2)$ such that the map $\mathcal{T}$ can be positive but is neither completely positive nor completely co-positive. For qubit maps this is equivalent to $\mathcal{T}_1, \mathcal{T}_2$ being not entanglement breaking.

An example for 2-LEA channels which are not entanglement breaking can be found in [SF12]. They study depolarizing qubit channels of the form

$$\mathcal{T}_q(X) = qX + (1-q)\frac{\mathrm{tr}(X)}{2} I$$

and they find that $\mathcal{T}_q \otimes \mathcal{T}_p$ is entanglement annihilating iff $pq \leq \frac{1}{3}$ for channels $\mathcal{T}_q$ and $\mathcal{T}_p$ of the above form. It is also known that $\mathcal{T}_q$ is entanglement breaking iff $q \leq \frac{1}{3}$. With this characterizations, one can prove the existence of entanglement annihilating channels that are not entanglement breaking.

We now take the map

$$\mathcal{T} = \mathcal{T}_p + \vartheta \circ \mathcal{T}_q$$

build from depolarizing channels. The conditions of this map to be neither completely positive nor completely co-positive can be stated via the Choi-matrices of the corresponding maps

$$\begin{aligned} C_{\mathcal{T}_p} + \mathrm{id} \otimes \vartheta C_{\mathcal{T}_q} &\ngeq 0 \\ C_{\mathcal{T}_q} + \mathrm{id} \otimes \vartheta C_{\mathcal{T}_p} &\ngeq 0\,. \end{aligned}$$

By explicitly calculating the Choi-matrices this means

$$\begin{pmatrix} \frac{2+p+q}{2} & 0 & 0 & p \\ 0 & \frac{2-p-q}{2} & q & 0 \\ 0 & q & \frac{2-p-q}{2} & 0 \\ p & 0 & 0 & \frac{2+p+q}{2} \end{pmatrix} \ngeq 0$$
$$\begin{pmatrix} \frac{2+p+q}{2} & 0 & 0 & q \\ 0 & \frac{2-p-q}{2} & p & 0 \\ 0 & p & \frac{2-p-q}{2} & 0 \\ q & 0 & 0 & \frac{2+p+q}{2} \end{pmatrix} \ngeq 0\,,$$

which corresponds to the conditions

$$\begin{aligned} 2 &< p + 3q \\ 2 &< q + 3p\,. \end{aligned}$$

In Figure 4.2 one can see the region where the above conditions are fulfilled together with the condition $pq \leq \frac{1}{3}$. This parameters all correspond by the above construction to positive, neither completely positive nor completely co-positive maps T for which $\mathcal{T} \otimes \mathcal{T}$ is positive. In

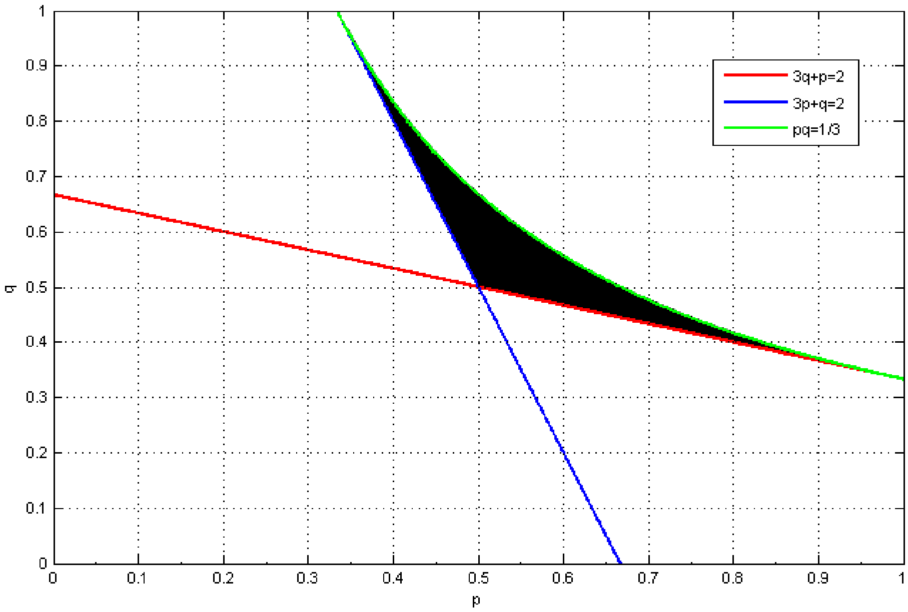

Figure 4.2.: Parameters p,q for two depolarizing channels $\mathcal{T}_p$ and $\mathcal{T}_q$ as building blocks for the map $\mathcal{T} = \mathcal{T}_p + \vartheta\mathcal{T}_q$. For $2 \leq p + 3q$ we have that $\mathcal{T}$ is not completely positive and for $2 \leq q + 3p$ that $\mathcal{T}$ is not completely co-positive. For $pq \leq \frac{1}{3}$ the map $\mathcal{T} \otimes \mathcal{T}$ is positive. In the black region all the inequalities are fulfilled together.

particular for $p = q = \frac{1}{\sqrt{3}}$ we get the map

$$\mathcal{T}(X) = \frac{1}{\sqrt{3}}\left(X + X^T\right) + \left(2 - \frac{2}{\sqrt{3}}\right)\frac{\operatorname{tr}(X)}{2}I$$

as an example for such a map.

This shows that the question of tensor-stable positivity is really a non-trivial one and that there might be non-trivial, i.e. neither completely positive nor completely co-positive, maps which are tensor-stable positive. Note that an $\infty-LEA$ map $\mathcal{R} \notin \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_n) \cap \mathfrak{coCP}(\mathfrak{M}_n, \mathfrak{M}_n)$ could give rise to such a map, via

$$\mathcal{T} = \mathcal{R} + \vartheta \circ \mathcal{R}\,.$$

Unfortunately no such map $\mathcal{R}$ is known at the time of writing. Note further that our method also works for channels $\mathcal{T}_1, \mathcal{T}_2$ such that $\mathcal{T}_i \otimes \mathcal{T}_j(X)$ is PPT for all quantum states X. But at the point of writing, we do not know of any example except the ones treated here.

5. Conclusion

We have studied possibilities to generalize the known criteria of entanglement distillation and the known bounds on quantum capacities. As we showed, these two problems are connected as a quantum channel has zero 2-way assisted capacity if and only if its Choi-matrix is not distillable. The known transposition bound for the quantum capacity is also a bound for the 2-way assisted capacity and one could try to generalize this bound directly. Furthermore we showed that the assumption of multiplicativity or antimultiplicativity seems too strong to provide new examples for linear maps, which generate distillation criteria. Also a milder assumption, where the map can be commuted with a completely positive map leaving behind a different completely positive map, strongly restricts the class of possible maps. Finally we proposed the possibility of tensor-stable positive maps, which are not completely positive. We showed how such maps would lead to bounds on the quantum capacity. Unfortunately the trivial case of completely co-positive map leads to bounds, which are worse than the known transposition bound. To obtain better bounds, we would need a non-trivial, i.e. neither completely positive nor completely co-positive tensor-stable positive map. At present we do not know whether such maps exist. We gave examples for 2-tensor-stable positive qubit maps, which are neither completely positive nor completely co-positive, but for higher dimensions no such example is known.

In our studies we assumed the maps to be linear in all cases. Nonlinear maps might provide new possibilities for distillation criteria and quantum capacity bounds, but are considerably harder to handle. Smith et al. [GS12] come to similar conclusions as there are currently two classes of channels with zero quantum capacity known. The first class corresponds to the transposition bound, which we also studied, and the second class to the cloning map, which is a nonphysical and nonlinear map, which could be implemented if certain channels had a non-zero quantum capacity.

Another direction of further study are general non-positive maps. Most of our results assumed the occurring maps to be either positive or, slightly more general, hermitian preserving. This is not directly needed in the known proofs of the transposition criterion. But again, such maps are considerably harder to study.

A. Norms of Matrix Maps

In this appendix we will introduce norms on spaces of matrix maps, which we use frequently in our argumentation. In particular we will introduce the cb-norm together with its dual the $\diamond-$norm and give proofs for properties of these norms.

We begin with the definition of two simpler norms, which will be used later to define the cb- and $\diamond-$norm.

Definition 25. ($\infty \to \infty$-*norm*, $1 \to 1$-*norm*)

For a map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ we define the norms

$$\|\mathcal{T}\|_{\infty\to\infty} := sup\{\|\mathcal{T}(X)\|_\infty : X \in \mathfrak{M}_n, \|X\|_\infty = 1\}$$

and

$$\|\mathcal{T}\|_{1\to 1} := sup\{\|\mathcal{T}(X)\|_1 : X \in \mathfrak{M}_n, \|X\|_1 = 1\} .$$

Here we used the usual ∞-, and trace-norm for matrices $X \in \mathfrak{M}_m$, which are defined as

$$\|X\|_\infty := s_1$$

and

$$\|X\|_1 := \sum_{i=1}^{m} s_i ,$$

where $(s_i)_{i=1}^m \subset \left(\mathbb{R}_0^+\right)^m$ denotes the singular values of X in decreasing order.

The following properties of these norms will lead to similar properties of the cb-norm and the $\diamond-$norm.

It follows directly from the duality of the trace- and the ∞-norm for matrices $X \in \mathfrak{M}_m$, that also the $1 \to 1-$norm and the $\infty \to \infty-$norm are duals to each other. Furthermore we get:

Lemma 11. (*Properties*)

For all maps $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ and $\mathcal{S} : \mathfrak{M}_k \to \mathfrak{M}_n$ we have

- $\|\mathcal{T}\|_{\infty\to\infty} = \|\mathcal{T}^*\|_{1\to 1}$ (*Duality*).
- $\|\mathcal{T} \circ \mathcal{S}\|_{\infty\to\infty} \leq \|\mathcal{T}\|_{\infty\to\infty}\|\mathcal{S}\|_{\infty\to\infty}$ (*Submultiplicativity*).

- $\|\mathcal{T} \circ \mathcal{S}\|_{1\to 1} \leq \|\mathcal{T}\|_{1\to 1}\|\mathcal{S}\|_{1\to 1}$ (*Submultiplicativity*).

The above properties are useful in calculations, but there is another property that we need in our argumentation. Namely that for every $\mathcal{T} \in \mathcal{P}(\mathfrak{M}_n, \mathfrak{M}_m)$ we have $\|\mathcal{T}\|_{\infty\to\infty} = \|\mathcal{T}(I)\|_{\infty}$. This is known as the Russo-Dye-Theorem and we will follow [Bha07] in the proofs.

Lemma 12. (*See [Bha07]*)
For $A, B \in \mathfrak{M}_n$ strictly positive we have

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0 \iff A \geq XB^{-1}X^*.$$

Proof. The similarity transformation

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \sim \begin{pmatrix} I & -XB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \begin{pmatrix} I & 0 \\ -B^{-1}X^* & I \end{pmatrix} = \begin{pmatrix} A - XB^{-1}X^* & 0 \\ 0 & B \end{pmatrix},$$

shows the lemma.

□

We are now able to prove Choi's inequality.

Lemma 13. (*Choi's Inequality. [Bha07]*)
For $\mathcal{T} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$ and unital then we have

$$\mathcal{T}(A)\,\mathcal{T}(A^*) \leq \mathcal{T}(A^*A)$$

and

$$\mathcal{T}(A^*)\,\mathcal{T}(A) \leq \mathcal{T}(A^*A)$$

for all normal $A \in \mathfrak{M}_n$.

Proof. We will only proof the first inequality as the second one works the same way. By the spectral decomposition we can write

$$A = \sum_i r_i \left|i\right\rangle \left\langle i\right| ,$$

with projectors $\{\left|i\right\rangle \left\langle i\right|\}$ such that $\sum_i \left|i\right\rangle \left\langle i\right| = I$ and $\{r_i\} \subset \mathbb{C}$. We also get

$$A^* = \sum_i \overline{r_i} \left|i\right\rangle \left\langle i\right|$$

and

$$A^*A = \sum_i |r_i|^2 |i\rangle \langle i| \ .$$

But this gives

$$\begin{pmatrix} \mathcal{T}(A^*A) & \mathcal{T}(A) \\ \mathcal{T}(A^*) & I \end{pmatrix} = \sum_i \begin{pmatrix} |r_i|^2 & r_i \\ \overline{\lambda_i} & 1 \end{pmatrix} \otimes \mathcal{T}(|i\rangle\langle i|) \ ,$$

which is positive and Lemma 12 finishes the proof.

□

We will now proceed and prove the Russo-Dye Theorem which is of great importance in the study of positive matrix maps.

Theorem 19. (*Russo-Dye Theorem. [Bha07]*)
If $\mathcal{T} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$ *we have*

$$\|\mathcal{T}\|_{\infty\to\infty} = \|\mathcal{T}(I)\|_\infty \ .$$

Proof. First assume that T is unital. For $\|A\|_\infty \leq 1$ consider

$$U_A = \begin{pmatrix} A & -(I - AA^*)^{\frac{1}{2}} \\ (I - A^*A)^{\frac{1}{2}} & A^* \end{pmatrix} \in \mathfrak{M}_{2n} \ .$$

A simple calculation shows that U_A is unitary.
Consider now the positive and unital map

$$\mathcal{T} \circ C_n : \mathfrak{M}_{2n} \to \mathfrak{M}_m \ ,$$

where $\mathcal{C}_n : \mathfrak{M}_{2n} \to \mathfrak{M}_n$ is defined as $\mathcal{C}_n(A) = (A)_{1:n,1:n}$. Applying Lemma 13 to $\mathcal{T} \circ \mathcal{C}_n$ and the unitary, hence normal, matrix U_A we get

$$(\mathcal{T} \circ \mathcal{C}_n(U_A))(\mathcal{T} \circ \mathcal{C}_n(U_A^*)) \leq \mathcal{T} \circ \mathcal{C}_n(I) \ .$$

Using the definition of U_A and $\mathcal{C}_n$ this gives

$$\mathcal{T}(A)\mathcal{T}(A^*) \leq I \ .$$

But this leads to $\|\mathcal{T}(A)\|_\infty \leq 1$ whenever $\|A\|_\infty \leq 1$, which shows $\|\mathcal{T}\|_{\infty\to\infty} = 1 = \|\mathcal{T}(I)\|_\infty$ for all unital $\mathcal{T} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$.

For an arbitrary $\mathcal{T} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$ assume that $\mathcal{T}(I)$ is invertible and consider the unital map $\widehat{\mathcal{T}} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$

$$\widehat{\mathcal{T}} = (\mathcal{T}(I))^{-1/2} \mathcal{T}(A) (\mathcal{T}(I))^{-1/2} \ .$$

By submultiplicativity of the norms we get

$$\|\mathcal{T}(A)\|_\infty \leq \|\mathcal{T}(I)\|_\infty \|\widehat{\mathcal{T}}(A)\|_\infty \leq \|\mathcal{T}(I)\|_\infty \|A\|_\infty \ ,$$

where we used the calculation for unital and positive $\widehat{\mathcal{T}}$. This means that in the case where $\mathcal{T}(I)$ is invertible, we get $\|\mathcal{T}\|_{\infty\to\infty} = \|\mathcal{T}(I)\|_\infty$. In the general case we apply a standard continuity argument on the family $T_\epsilon(A) = \mathcal{T}(A) + \epsilon I$, for which $T_\epsilon(I)$ is invertible, which finishes the proof.

□

By duality we immediately get the following:

Theorem 20. *If $\mathcal{T} \in \mathfrak{P}(\mathfrak{M}_n, \mathfrak{M}_m)$ we have*

$$\|\mathcal{T}\|_{1\to1} = \|\mathcal{T}^*(I)\|_1 \ .$$

This shows that for the $\infty \to \infty$-norm the positive unital maps and for the $1 \to 1$-norm the positive trace-preserving maps have norm 1. Surprisingly also the converse holds true:

Theorem 21. (*[Bha07]*)
For $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ linear and unital we have

$$[\|\mathcal{T}\|_{\infty\to\infty} = 1] \Rightarrow \mathcal{T} \ \textit{positive}\,.$$

Proof. Assume first that $m = 1$, i.e. $\mathcal{T}$ is a linear, unital functional on $\mathfrak{M}_n$.

Take $A \in \mathfrak{M}_n^+$ and we denote $a = \min(\text{spec}(A)) \in \mathbb{R}_0^+$, $b = \max(\text{spec}(A)) \in \mathbb{R}_0^+$.

Assuming $\mathbb{R} \ni \mathcal{T}(A) \notin [a, b]$, then there is a disk $\mathfrak{D}_r(z_0)$ with radius $r \in \mathbb{R}^+$ and center $z_0 \in \mathbb{C}$ such that $\mathcal{T}(A) \notin \mathfrak{D}_r(z_0)$, but

$$\text{spec}(A) \subset [a, b] \subset \mathfrak{D}_r(z_0) \ .$$

From the latter property, we obtain

$$\text{spec}(A - z_0) \subset \mathfrak{D}_r(0)$$

and therefore $\|A - z_0\|_\infty \leq r$.

But this leads, using unitality and the norm property, to

$$|\mathcal{T}(A) - z_0| = |\mathcal{T}(A - z_0 I_n)| \leq \|\mathcal{T}\|_{\infty\to\infty} \|A - z_0\|_\infty \leq r$$

which contradicts $\mathcal{T}(A) \notin \mathfrak{D}_r(z_0)$ and shows the assumption for a functional.

For arbitrary $m \in \mathbb{N}$, we define the functional $\widehat{\mathcal{T}_x} : \mathfrak{M}_n \to \mathbb{C}$, for a vector $|x\rangle \in C^m$, via

$$\widehat{\mathcal{T}_x}(A) = \langle x| \mathcal{T}(A) |x\rangle \ .$$

For every $|x\rangle \in C^m$ the above functional is linear and unital. Furthermore we have $\|\widehat{\mathcal{T}_x}\|_{\infty\to\infty} \leq \|\mathcal{T}\|_{\infty\to\infty} \leq 1$ according to the assumption. The above calculation shows that $\widehat{\mathcal{T}_x}$ is positive for every $|x\rangle \in C^m$, but this is exactly the condition for positivity of $\mathcal{T}$.

□

Again we can easily show the dual version:

Theorem 22. *For $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ linear and trace-preserving we have*

$$[\|\mathcal{T}\|_{1\to1} = 1] \Rightarrow \mathcal{T} \text{ positive}.$$

We want to have useful norms for maps arising in quantum information theory. One property that such a norm should have, is stability under the transformation $\mathcal{T} \mapsto \mathrm{id}_k \otimes \mathcal{T}$, i.e. tensoring of an identity of arbitrary dimension. It should not matter if we take an arbitrary environment and define a quantum channel, which acts trivially on the environment and as $\mathcal{T}$ on the system. That this is not the case for the above norms can be seen by considering the transposition map. We introduce the following stabilized versions, which are called the cb-norm and its dual the $\diamond$-norm. These norms are invariant under tensoring of an identity.

Definition 26. (*cb-norm,* $\diamond$*-norm*)

*For a map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$ we define the **cb-Norm** as*

$$\|\mathcal{T}\|_{cb} = sup_{k\in\mathbb{N}} \|id_k \otimes \mathcal{T}\|_{\infty\to\infty}$$

*and the **$\diamond$-Norm** as*

$$\|\mathcal{T}\|_{\diamond} = sup_{k\in\mathbb{N}} \|id_k \otimes \mathcal{T}\|_{1\to1} \ .$$

These norms have many nice properties, which we will sketch in the following. First we can establish the connection between the two norms. We have the following, which mostly follows from the properties of the $\infty \to \infty$- and $1 \to 1$-norm :

Lemma 14. (*Properties*)
For all maps $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$, $\mathcal{S} : \mathfrak{M}_k \to \mathfrak{M}_n$ *and* $\mathcal{R} : \mathfrak{M}_k \to \mathfrak{M}_l$ *we have*

- $\|\mathcal{T}\|_{cb} = \|\mathcal{T}^*\|_\diamond$ (*Duality*).
- $\|\mathcal{T} \circ \mathcal{S}\|_{cb} \leq \|\mathcal{T}\|_{cb}\|\mathcal{S}\|_{cb}$ (*Submultiplicativity*).
- $\|\mathcal{T} \circ \mathcal{S}\|_\diamond \leq \|\mathcal{T}\|_\diamond\|\mathcal{S}\|_\diamond$ (*Submultiplicativity*).
- $\|\mathcal{T} \otimes \mathcal{R}\|_{cb} = \|\mathcal{T}\|_{cb}\|\mathcal{R}\|_{cb}$ (*Cross-norm Property*).
- $\|\mathcal{T} \otimes \mathcal{R}\|_\diamond = \|\mathcal{T}\|_\diamond\|\mathcal{R}\|_\diamond$ (*Cross-norm Property*).

From the properties of the $\infty \to \infty-$ and the $1 \to 1-$norm, namely Theorems 19, 20, 21 and 22, we get:

Theorem 23. *If* $\mathcal{T} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_m)$ *we have*

$$\|\mathcal{T}\|_{cb} = \|\mathcal{T}\|_{\infty\to\infty} = \|\mathcal{T}(I)\|_\infty$$

and

$$\|\mathcal{T}\|_\diamond = \|\mathcal{T}\|_{1\to 1} = \|\mathcal{T}^*(I)\|_1 .$$

Proof. As $\mathcal{T} \in \mathfrak{CP}(\mathfrak{M}_n, \mathfrak{M}_m)$ means, that $\mathrm{id}_k \otimes \mathcal{T}$ is positive for all $k \in \mathbb{N}$, we get the Theorem immediately from the Russo-Dye Theorem, Theorem 19, and its corollaries.

□

To conclude this appendix, we will state one important inequality for the cb-norm. The following bound is due to Smith

Theorem 24. (*See [Pau03]*)
For a map $\mathcal{T} : \mathfrak{M}_n \to \mathfrak{M}_m$, *we have*

$$\|\mathcal{T}\|_{cb} = \|id_m \otimes \mathcal{T}\|_{\infty\to\infty} \leq m\|\mathcal{T}\|_\infty .$$

B. Projective Geometry

In this Appendix we will review some facts about projective geometry, that are important for the theory of linear preserver problems. We will start with the definition of projective space and the notion of collinear maps between projective spaces to lay the background for the fundamental theorem of projective geometry. We will mostly follow the introduction in [AO06].

Definition 27. (*Projective space, projective points. [AO06]*) *Let $\mathfrak{V}$ be a vector space over a skew field $\mathfrak{K}$. We define*

$$\mathfrak{P}(\mathfrak{V}) := \{[v] : |v\rangle \in \mathfrak{V} \setminus \{0\}\}$$

and call this set the ***projective space*** *associated to $\mathfrak{V}$. Here $[v]$ denotes the 1-dimensional subspace of $\mathfrak{V}$ generated by $|v\rangle \in \mathfrak{V} \setminus \{0\}$, also called* ***projective points*** *in this context.*

It will be convenient to identify also $0 \in \mathfrak{V}$, with a point adjoined to the projective space associated to $\mathfrak{V}$. This element will be called the ***nopoint*** *and is denoted by $[0]$. With this convention, we define*

$$\mathfrak{P}_0(\mathfrak{V}) := \{[v] : |v\rangle \in \mathfrak{V}\}$$

where we adjoined the nopoint to the projective space.

In the above definition we introduced the projective space $\mathfrak{P}(\mathfrak{V})$ as the set containing all 1-dimensional subspaces of $\mathfrak{V}$. The same construction can be done for subspaces of higher dimensions. The **projective geometry** associated to the vector space $\mathfrak{V}$, denoted by $\mathfrak{PG}(\mathfrak{V})$, is the set containing all subspaces of $\mathfrak{V}$. We will call its elements, i.e. subspaces of a certain dimension, **projective subspaces**. The **projective dimension** of such a projective subspace corresponding to a d-dimensional subspace of $\mathfrak{V}$ will be $d-1$.

This means, that we call lines in the vector space **projective points**, planes in the vector space **projective lines**, and so on.

We will now define an operation to combine projective subspaces to form new projective subspaces.

Definition 28. (*Join. [AO06]*)

For $A, B \in \mathfrak{PG}(\mathfrak{V})$ we define the ***join*** *as*

$$A \vee B = span(v \in A \cup B) \ ,$$

i.e. the minimal subspace of $\mathfrak{V}$ containing both A and B.

Now we define a class of maps on the projective spaces, which preserve the join operation, that we defined above. Here it can be proved sufficient to consider projective points in the following definition [AO06].

Definition 29. (*Collinear maps. [AO06]*)
Let $\mathfrak{P}_0(V_1), \mathfrak{P}_0(V_2)$ be two projective spaces. A map

$$\mathcal{G} : \mathfrak{P}_0(V_1) \to \mathfrak{P}_0(V_2)$$

*is called **collinear** if it fulfills*

1. $\mathcal{G}([0]) = [0]$.
2. $\mathcal{G}([v] \vee [w]) = \mathcal{G}([v]) \vee \mathcal{G}([w])$.

*A bijective collinear map is also called a **collineation**.*

A natural question is, what kind of maps are collinear. The answer to this question is called the fundamental theorem of projective geometry, which we are going to introduce next. The property of a map being collinear will turn out quite strong.

Before stating this theorem we have to state some more definitions.

Definition 30. (*Semilinear maps. [AO06]*)
*A map $f : V_1 \to V_2$ between vector spaces V_1, V_2 over skew-fields $\mathfrak{K}_1, \mathfrak{K}_2$ is called **semilinear** if there is an homomorphism between skew-fields $\sigma : \mathfrak{K}_1 \to \mathfrak{K}_2$, such that*

$$f(a|v\rangle + b|w\rangle) = \sigma(a) f(|v\rangle) + \sigma(b) f(|w\rangle), \forall |v\rangle, |w\rangle \in V_1 .$$

One important property of semilinear maps in this context is, that they lead to collinear maps via the following construction:

Proposition 3. *Let $f : V_1 \to V_2$ be semilinear. Then the map*

$$\mathfrak{P}f : \mathfrak{P}(V_1) \setminus \mathfrak{P}(ker(f)) \to \mathfrak{P}(V_2)$$
$$\mathfrak{P}f([v]) = [f(|v\rangle)]$$

*is a collinear map between the projective spaces $\mathfrak{P}(V_1)$ and $\mathfrak{P}(V_2)$. We call $\mathfrak{P}f$ the collinear map **induced by f**.*

Proof. Trivial □

Now we are able to formulate the fundamental theorem of projective geometry, which states that every bijective collinear map between projective spaces is of the above form, i.e. induced by an semilinear map.

Theorem 25. (*Fundamental Theorem of Projective Geometry. [AO06]*)
Let $\mathfrak{V}_1, \mathfrak{V}_2$ *be vector spaces over skew-fields* $\mathfrak{K}_1, \mathfrak{K}_2$ *and*

$$\mathcal{G} : \mathfrak{P}_0\left(V_1\right) \rightarrow \mathfrak{P}_0\left(V_2\right)$$

a bijective collinear map, i.e. a collineation. Then there is a bijective semilinear map $f : \mathfrak{V}_1 \rightarrow \mathfrak{V}_2$ *that induces* $\mathcal{G}$*, i.e. we have* $\mathcal{G} = \mathfrak{P} f$*, as defined in Proposition 3.*

Proof. See [AO06]. □

C. Differential Geometry

In this Appendix we will provide some results from differential geometry, which are needed for the proof of Theorem 7. We will start with the basic definitions, which we then apply to the unitary orbit in which we are interested in. We will follow some of the argumentation in [Bha97]

Definition 31. (*Smooth Embedded Manifold*)

A subset $\mathfrak{M} \subseteq \mathbb{R}^n$ is called a ***smooth k-dimensional embedded manifold*** *if for every $x \in \mathfrak{M}$ there is an environment $\mathfrak{E} \subset \mathbb{R}^n$ of x and a $\mathcal{C}^\infty-$map*

$$F : \mathfrak{E} \to \mathbb{R}^{n-k} \ ,$$

such that $\mathfrak{E} \cap \mathfrak{M} = F^{-1}(0)$ and rank$(J_x F) = n - k$.

The unitary orbit of $A \in \mathfrak{H}_n$, defined as

$$\begin{aligned} \mathfrak{U}_A &= \{B \in \mathfrak{H}_n : B \sim_U A\} \\ &= \{UAU^\dagger : U \text{ unitary}\} \end{aligned}$$

is indeed a smooth embedded manifold in $\mathfrak{H}_n \simeq \mathbb{R}^{n^2}$.

To see this consider the function $F : \mathfrak{H}_n \to \mathbb{R}$ defined by

$$F(X) = \|\lambda(X) - \lambda(A)\|_2^2 \ ,$$

where $\lambda(X) \in \mathbb{R}^n$ denotes the spectrum of $X \in \mathfrak{H}_n$ in decreasing order. It is easy to see, that this function fulfills the requirements stated in the Definition and that $F(\mathcal{U}_A) = \{0\}$, which shows the assumption.

Next we need the definition of tangent space of a manifold.

Definition 32. (*Tangent space*)

Let $\mathfrak{M} \subseteq \mathbb{R}^n$ be an embedded manifold. We define the ***tangent space*** *of the manifold $\mathfrak{M}$ at point $x \in \mathfrak{M}$ as*

$$\mathfrak{T}_x\mathfrak{M} := \{\gamma'(0) \in \mathbb{R}^n | \ \gamma : (-\epsilon, \epsilon) \to \mathfrak{M} \textit{ differentiable curve with } \gamma(0) = x\} \ .$$

We will now compute the tangent space $\mathfrak{T}_A\mathcal{U}_A$ to the unitary orbit in the point A. Note that in this setting the differentiable curves in the definition of the tangent space have the form:

$$\gamma(t) = \mathcal{U}(t)\, A\mathcal{U}(t)^\dagger$$
$$\text{with a function } \mathcal{U} : [-\epsilon, \epsilon] \to \mathfrak{U}_n, \mathcal{U}(0) = I\ .$$

By unitarity we have $\mathcal{U}(t)\mathcal{U}(t)^\dagger = I$ and by differentiating this expression at $t = 0$ we get

$$\mathcal{U}'(0) = -\mathcal{U}'(0)^\dagger\ ,$$

which means that $\mathcal{U}'(0)$ is skew-hermitian.

Consider the curve

$$\mathcal{E}(t) = \exp(tS)$$

for some skew-hermitian matrix S. This curve lies in $\mathfrak{U}_n$ and fulfills $\mathcal{E}(0) = I$ and $\mathcal{E}'(0) = S$. Thus $\mathcal{E}$ is a first order approximant to the curve $\mathcal{U}$, defined above, with $\mathcal{U}'(0) = S$, and therefore both maps describe the same tangent vector S.

Now we can compute the tangent space $\mathfrak{T}_A\mathfrak{U}_A$ by considering the curves $\mathcal{E}$ defined above

$$\gamma(t) = \exp(tS)\, A\exp(-tS)$$
$$\text{with } S \text{ skew-hermitian}\ .$$

By differentiation we obtain

$$\mathfrak{T}_A\mathfrak{U}_A = \{[A, S] : S \text{ skew-hermitian}\}\ ,$$

which we wanted to show in this Appendix.

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