Spectral analysis of a model for quantum friction

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Joint work with S. De Bièvre and B. Schubnel
The model

Main results

Ingredients of the proof
Spectral analysis of a model for quantum friction

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The model

Classical Hamiltonian model
Quantum model

Main results

Ingredients of the proof

The model
Linear friction

Linear Friction

Many classical systems – e.g. an electron in a metal, a particle in a viscous medium – obey an effective equation of motion of the form

\[ m\ddot{q}(t) = -\gamma \dot{q}(t) - \nabla V(q(t)) \]  

(1)

where

- \( q(t) \in \mathbb{R}^d \) is the position of the system
- \( m \) is the mass of the system
- \( \gamma > 0 \) is the friction coefficient
- \( V \) is an external potential

In particular, \( V = 0 \implies \dot{q}(t) \) converges exponentially fast to 0

Interaction with the environment

- (1) = effective equation of motion
- Friction force due to the energy lost by the system, transferred to the environment
- More fundamental approach: model describing both the system and its environment with total energy conserved
A classical Hamiltonian model

[Bruneau, De Bièvre, 2002]

Particle of position $q(t) \in \mathbb{R}^d$ (mass $m = 1$, no external potential) coupled to independent scalar vibration fields $\psi(x, y, t) \in \mathbb{R}$ at each point $x \in \mathbb{R}^d$ ($y \in \mathbb{R}^3$ accounts for the position variable in the “propagation space” of the fields)

Equations of motion

$$\frac{\partial^2}{\partial t^2} \psi(x, y, t) - c^2 \Delta_y \psi(x, y, t) = -\rho_1(x - q(t))\rho_2(y)$$

$$\ddot{q}(t) = -\int_{\mathbb{R}^{d+3}} \rho_1(x - q(t))\rho_2(y)(\nabla_x \psi)(x, y, t) \, dx dy$$
A classical Hamiltonian model II

Equations of motion

\[
\partial_t^2 \psi(x, y, t) - c^2 \Delta_y \psi(x, y, t) = -\rho_1(x - q(t))\rho_2(y) \\
\ddot{q}(t) = -\int_{\mathbb{R}^{d+3}} \rho_1(x - q(t))\rho_2(y)(\nabla_x \psi)(x, y, t) \, dx \, dy
\]

Should be compared with

Classical Nelson model

Classical particle coupled to a scalar wave field

\[
\partial_t^2 \psi(x, t) - c^2 \Delta_x \psi(x, t) = -\rho_1(x - q(t)) \\
\ddot{q}(t) = -\int_{\mathbb{R}^d} \rho_1(x - q(t))(\nabla_x \psi)(x, t) \, dx
\]

Other related classical models

See [Komech, Spohn, 1998], [Komech, Kunze, Spohn, 1998]
A classical Hamiltonian model III

Equations of motion

\[ \begin{align*}
\partial_t^2 \psi(x, y, t) - c^2 \Delta_y \psi(x, y, t) &= -\rho_1(x - q(t))\rho_2(y) \\
\ddot{q}(t) &= -\int_{\mathbb{R}^{d+3}} \rho_1(x - q(t))\rho_2(y)(\nabla_x \psi)(x, y, t) \, dx \, dy
\end{align*} \]

Assumptions

- \( \rho_1 \in S(\mathbb{R}^d) \), positive, radial
- \( \rho_2 \in S(\mathbb{R}^3) \), positive, radial and \( \hat{\rho}_2(k) \neq 0 \) \( \forall k \in \mathbb{R}^3 \)

Results [Bruneau, De Bièvre 2002]

For a large class of initial data, and for \( c \) large enough, the particle stops exponentially fast,

\[ |q(t) - q_\infty| \leq Ce^{-\gamma t}, \quad t \geq 0 \]
### Hilbert space

**Hilbert space for the particle and the field**

\[ \mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}_s(L^2(\mathbb{R}^{d+3})) \]

### Symmetric Fock space

- 
  \[ \mathcal{F}_s(L^2(\mathbb{R}^{d+3})) = \bigoplus_{n \geq 0} \mathcal{F}_s^{(n)} \]
  
  where
  \[ \mathcal{F}_s^{(0)} := \mathbb{C}, \quad \mathcal{F}_s^{(n)} := L^2_s(\mathbb{R}^{(d+3)n}) \]

- Creation and annihilation operators denoted by \( a^*(\xi, k), a(\xi, k) \) (momentum variables) satisfy the canonical commutation relations
  \[
  [a(\xi, k), a^*(\xi', k')] = \delta(\xi - \xi')\delta(k - k'), \\
  [a^#(\xi, k), a^#(\xi', k')] = 0
  \]
Total Hamiltonian acting on $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}_s(L^2(\mathbb{R}^{d+3}))$

$$H := \frac{-\Delta_q}{2} \otimes 1 + 1 \otimes H_f + g H_I,$$

where

- Hamiltonian for the particle: $-\Delta_q/2$
- Hamiltonian for the field:
  $$H_f = \int_{\mathbb{R}^{d+3}} |k| a(\xi, k) a(\xi, k) d\xi dk$$
- Interaction Hamiltonian:
  $$H_I := \int_{\mathbb{R}^{d+3}} (e^{-iq \cdot \xi} |k|^\mu \hat{\rho}_1(|\xi|) \hat{\rho}_2(|k|) a(\xi, k) + e^{iq \cdot \xi} |k|^\mu \hat{\rho}_1(|\xi|) \hat{\rho}_2(|k|) a^*(\xi, k)) d\xi dk$$

- $g \in \mathbb{R}$: coupling constant
- $\mu \geq -1/2$: infrared regularization
- $\rho_1 \in \mathcal{S}(\mathbb{R}^d)$, $\rho_2 \in \mathcal{S}(\mathbb{R}^3)$
### Self-adjointness

For all \( g \in \mathbb{R} \) and \( \mu > -1 \), \( H \) is a self-adjoint operator with domain

\[
D(H) = D(H_0),
\]

where \( H_0 := H|_{g=0} \)

### Translation invariance

- Let

\[
P_f = \int_{\mathbb{R}^{d+3}} \xi a^*(\xi, k)a(\xi, k)d\xi dk
\]

Then

\[
\left[ (-i\nabla_q \otimes \mathbb{1} + \mathbb{1} \otimes P_f)_j, H \right] = 0, \quad j = 1, \ldots, d
\]

- Unitary transformation \( U : \mathcal{H} \to \int_{\mathbb{R}^d} \mathcal{H}_p dp, \mathcal{H}_p = \mathcal{F}_s(L^2(\mathbb{R}^{d+3})), \) such that

\[
UHU^* = \int_{\mathbb{R}^d} H(p) dp
\]
The fiber Hamiltonian

Fiber Hamiltonian acting on $\mathcal{F}_s(L^2(\mathbb{R}^{d+3}))$

- For all $p \in \mathbb{R}^d$,
  \[ H(p) := (p - P_f)^2 / 2 + H_f + g H_{I,0}, \]

- Interaction Hamiltonian at fixed total momentum:
  \[ H_{I,0} := \int_{\mathbb{R}^{d+3}} |k|^{\mu} (\hat{\rho}_1(|\xi|)\hat{\rho}_2(|k|)a^* (\xi, k) + \hat{\rho}_1(|\xi|)\hat{\rho}_2(|k|)a (\xi, k)) \, d\xi \, dk \]

- For all $p \in \mathbb{R}^d$, $g \in \mathbb{R}$ and $\mu > -1$, $H(p)$ is a self-adjoint operator with domain \[ \mathcal{D}(H(p)) = \mathcal{D}(H_f) \cap \mathcal{D}(P_f^2) \]

Spectrum of the non-interacting Hamiltonian

\[ \sigma(H_0(p)) = \sigma_{ess}(H_0(p)) = \sigma_{ac}(H_0(p)) = [0, \infty), \]
\[ \sigma_{pp}(H_0(p)) = \{p^2/2\}, \quad \sigma_{sc}(H_0(p)) = \emptyset \]

Moreover $p^2/2$ is a simple eigenvalue associated to the vacuum $\Omega \in \mathcal{F}_s(L^2(\mathbb{R}^{d+3}))$
Main results
Theorem [De Bièvre, Faupin, Schubnel]

i) Suppose that $\mu > -1$. For all $g \in \mathbb{R}$, there exists $E_g \leq 0$ such that
\[ \sigma(H(p)) = \sigma_{\text{ess}}(H(p)) = [E_g, \infty), \]
for all $p \in \mathbb{R}^d$. In particular, $E_g = \inf \sigma(H(p))$ does not depend on $p$.

ii) Suppose that $\mu > -1/2$. There exists $g_c = g_c(\mu) > 0$ such that, for all $0 \leq |g| \leq g_c$,
\[ H(0) \text{ admits a unique ground state}, \]
namely $E_g$ is a simple eigenvalue of $H(0)$.

ii') Suppose that $-1 < \mu \leq -1/2$ and that $\hat{\rho}_1(0) \neq 0$, $\hat{\rho}_2(0) \neq 0$. For all $p \in \mathbb{R}^d$ and $g \in \mathbb{R}$,
\[ H(p) \text{ does not have a ground state} \]

iii) Suppose that $\mu > 1/2$. There exists $g_c = g_c(\mu) > 0$ such that, for all $0 \leq |g| \leq g_c$,
\[ \sigma_{pp}(H(0)) = \{E_g\}, \quad \sigma_{ac}(H(0)) = [E_g, \infty), \quad \sigma_{sc}(H(0)) = \emptyset. \]
Suppose in addition that $\hat{\rho}_1$ and $\hat{\rho}_2$ do not vanish and let $\nu_1, \nu_2$ be such that $0 < \nu_1 < \nu_2$. Then there exists $g_c = g_c(\mu, \nu_1, \nu_2) > 0$ such that, for all $0 < |g| \leq g_c$ and $p \in \mathbb{R}^d$, $|p| \in (\nu_1, \nu_2)$,
\[ \sigma_{pp}(H(p)) = \emptyset, \quad \sigma_{ac}(H(p)) = [E_g, \infty), \quad \sigma_{sc}(H(p)) = \emptyset. \]
In particular, for $|p| \in (\nu_1, \nu_2)$, $H(p)$ does not have a ground state and the unperturbed eigenvalue $p^2/2$ disappears as the coupling is turned on.
Main results

If the coupling constant $g = 0$, $\inf \sigma(H_0(p)) = 0$ for all $p$; $p^2/2$ is a simple eigenvalue of $H(p)$.

If the coupling constant $g \neq 0$, $\inf \sigma(H(p)) = E_g < 0$ for all $p$; $E_g$ is an eigenvalue if and only if $p = 0$; If $p \neq 0$, the spectrum is purely absolutely continuous.

Figure: Grey : absolutely continuous spectrum
Ingredients of the proof
Location of the spectrum

**Theorem**

Let $\mu > -1$ and $g \in \mathbb{R}$. There exists $E_g \leq 0$ such that

$$\sigma(H(p)) = [E_g, \infty),$$

for all $p \in \mathbb{R}^d$

**Idea**

- **Localization techniques** ([Derezinski, Gérard, 1998])

- General idea: To any state $\varphi$ with total momentum $p$, sufficiently localized in $x$-space, we can add a one-particle state $a^*(f)\Omega$, with $f$ localized near infinity in $x$-space, such that $a^*(f)\Omega$ has a momentum close to $\xi = -p$ and an energy close to $|k| = 0$. Then $a^*(f)\varphi$ (which can be defined in a proper sense) has an energy arbitrary close to $\varphi$ and a momentum arbitrary close to 0.

$$\implies \inf \sigma(H(0)) \leq \inf \sigma(H(p))$$

- Difficulty: estimate localization errors, in particular control the number of particles in the minimizing sequence
## Existence of a ground state for $H(0)$

### Theorem

Let $\mu > -1/2$. There is $g_c > 0$ such that for all $|g| \leq g_c$, $H(0)$ has a ground state.

### Idea

- **Spectral renormalization group** ([Bach, Fröhlich, Sigal 1998])
- **Iterative version introduced in** ([Ballesteros, Faupin, Fröhlich, Schubnel 2015])
- **Important new feature**: control first and second derivatives of Wick monomial kernels. Use rotation invariance.

### Remark

- [Pizzo 2003] : iterative perturbation theory (not applicable here)
**Infrared problem : absence of ground state for \( \mu \leq -1/2 \)**

**Theorem**

Suppose that \(-1 < \mu \leq -1/2\) and that \(\hat{\rho}_1(0) \neq 0, \hat{\rho}_2(0) \neq 0\). For all \(p \in \mathbb{R}^d\) and \(g \in \mathbb{R}\),

\[
H(p) \text{ does not have a ground state}
\]

**Idea**

- Argument by contradiction
- Use the pull-through formula
- Adapt a simple argument of [Derezinski, Gérard 2004]
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The model

Main results

Ingredients of the proof

Absolutely continuous spectrum, Local decay

Theorem

Suppose that $\mu > 1/2$. There exists $g_c > 0$ such that, for all $|g| \leq g_c$ and $p \in \mathbb{R}^d$, the following holds: Let $J \subset [E_g, \infty)$ be a compact interval such that $\sigma_{pp}(H(p)) \cap J = \emptyset$. Then

$$\sup_{z \in S} \| \langle A \rangle^{-s}(H(p) - z)^{-1}\langle A \rangle^{-s} \| < \infty,$$

for any $1/2 < s \leq 1$, with $A = d\Gamma(ik \cdot \nabla k/|k| + h.c.)$, $\langle A \rangle = (1 + A^*A)^{1/2}$ and

$$S = \{z \in \mathbb{C}, \text{Re}(z) \in J, 0 < |\text{Im}(z)| \leq 1\}.$$

In particular, the spectrum of $H(p)$ in $J$ is purely absolutely continuous. Moreover,

$$\| \langle A \rangle^{-s} e^{-itH(p)} \chi(H(p)) \langle A \rangle^{-s} \| \lesssim t^{-s+\frac{1}{2}}, \quad t \to \infty,$$

for any $1/2 < s \leq 1$ and $\chi \in C^\infty_0(J; \mathbb{R})$

Idea

- Mourre’s commutator method [Mourre 1981]
- Extension with a non self-adjoint conjugate operator, and a first commutator not controllable by the Hamiltonian [Georgescu, Gérard, Møller 2004]
Absence of eigenvalues for $H(p)$, $p \neq 0$, $g \neq 0$

**Theorem**

Let $\mu > 1/2$ and $\nu_1, \nu_2$ be such that $0 < \nu_1 < \nu_2$. There exists $g_c = g_c(\mu, \nu_1, \nu_2) > 0$ such that, for all $0 < |g| \leq g_c$ and $p \in \mathbb{R}^d$, $|p| \in (\nu_1, \nu_2)$,

$$\sigma_{pp}(H(p)) = \emptyset$$

**Idea**

- Moure’s commutator method [Georgescu, Gérard, Møller 2004]
- **Fermi Golden Rule** criterion ([Hunziker, Sigal 2000], [Faupin, Møller, Skibsted 2011])

$$\Pi_\Omega H_{l,0} \text{Im}((H_0(p) - p^2 - i0^+)^{-1}\Pi_\Omega)H_{l,0}\Pi_\Omega \geq c(p)\Pi_\Omega,$$

where $\Pi_\Omega$ is the projection onto the Fock vacuum and $\tilde{\Pi}_\Omega := 1 - \Pi_\Omega$
Thank you!