Algebraic Quantum Field Theory
Homework Sheet 4

Problem 1. Show that in the sense of quadratic forms on $D \times D$, where

$$D = \{ \Psi \in \Gamma_{\text{fin}}(\mathfrak{h}) \mid \Psi^{(n)} \in S(\mathbb{R}^n) \text{ for all } n \},$$

we have the following representations for the (free) Hamiltonian $H := d\Gamma(\mu_m)$:

$$d\Gamma(\mu_m(p)) = \int d^d k \mu_m(k)a^*(k)a(k)$$

$$= \frac{1}{2} \int d^d x \left( : \pi^2_{\mu_m}(x) : + : \nabla \varphi^2_{\mu_m}(x) : + m^2 : \varphi^2_{\mu_m}(x) : \right).$$

(2)

Here $\mu_m(p) = \sqrt{p^2 + m^2}$ and

$$\varphi_{\mu_m}(x) := \frac{1}{(2\pi)^{d/2}} \int \frac{d^d k}{\sqrt{2\mu_m(k)}} (e^{-ikx}a^*(k) + e^{ikx}a(k)), \quad \text{and}$$

$$\pi_{\mu_m}(x) := \frac{i}{(2\pi)^{d/2}} \int d^d k \sqrt{\frac{\mu_m(k)}{2}} (e^{-ikx}a^*(k) - e^{ikx}a(k)).$$

The Wick ordering $:\cdots :$ means shifting creation operators to the left and annihilation operators to the right, ignoring the commutators. For example

$$: (a^*(k_1)a^*(k_2) + a^*(k_1)a(k_2) + a(k_1)a^*(k_2) + a(k_1)a(k_2)) :$$

$$= a^*(k_1)a^*(k_2) + a^*(k_1)a(k_2) + a^*(k_2)a(k_1) + a(k_1)a(k_2).$$

(5)

(6)

Solution: Define

$$\phi_h(x) := \frac{1}{(2\pi)^{d/2}} \int d^d k (h(k)e^{-ikx}a^*(k) + \overline{h(k)}e^{ikx}a(k)).$$

(7)

For $h_1(k) := \frac{m}{\sqrt{2\mu_m(k)}}$ we get $m\varphi(x)$, for $h_2(k) := i\sqrt{\frac{\mu_m(k)}{2}}$ we get $\pi(x)$ for $h_3(k) := -i\sqrt{\frac{k}{2\mu_m(k)}}$ we get $\nabla \varphi(x)$. We consider matrix elements $\langle \psi_1, : \phi_h(x)^2 : \psi_2 \rangle$, where $\psi_1, \psi_2 \in D$. This gives rise to expressions $\langle \psi_1, : a^*(k_1)a(k_2) : \psi_2 \rangle$ etc. By definition of $a(k)$ and $D$ these expressions are Schwartz class functions of $k_1, k_2$. This observation justifies the manipulations below. We do not write $\psi_1, \psi_2$ explicitly, but they are always understood. We compute:
Thus we have that
\[ \int d^d x : \phi_h(x)^2 := \frac{1}{(2\pi)^d} \int d^d x \int d^d k_1 d^d k_2 : (h(k_1)e^{-ik_1 x} a^*(k_1) + \bar{h}(k_1)e^{ik_1 x} a(k_1)) \times (h(k_2)e^{-ik_2 x} a^*(k_2) + \bar{h}(k_2)e^{ik_2 x} a(k_2)) : \] (8)
This gives
\[ \int d^d x : \phi_h(x)^2 : = \int d^d k_1 d^d k_2 \left( a^*(k_1)a^*(k_2)h(k_1)h(k_2)\delta(k_1 + k_2) + a^*(k_1)a(k_2)h(k_1)\bar{h}(k_2)\delta(k_1 - k_2) + a^*(k_2)a(k_1)\bar{h}(k_1)h(k_2)\delta(k_1 - k_2) + a(k_1)a(k_2)h(k_1)\bar{h}(k_2)\delta(k_1 + k_2) \right). \] (9)
Consequently:
\[ \int d^d x : \phi_h(x)^2 : = \int d^d k \left( a^*(k)a^*(-k)h(k)h(-k) + 2a^*(k)a(k)|h(k)|^2 + a(k)a(-k)\bar{h}(k)\bar{h}(-k) \right) \]
\[ = \int d^d k 2a^*(k)a(k)|h(k)|^2 + \int d^d k \left( a^*(k)a^*(-k)h(k)h(-k) + \text{h.c.} \right) \] (11)
The last expression on the r.h.s. of (2), let's call it \( H_3 \), is given by
\[ H_3 = \frac{1}{2} \sum_{i=1}^{3} \int d^d x : \phi_{h_i}(x)^2 : \] (12)
But we have
\[ 2 \sum_{i=1}^{3} |h_i(k)|^2 = 2 \left( \frac{m^2}{2\mu_m(k)} + \frac{\mu_m(k)}{2} + \frac{k^2}{2\mu_m(k)} \right) = \mu_m(k), \]
\[ \sum_{i=1}^{3} h_i(k)h_i(-k) = \frac{m^2}{2\mu_m(k)} - \frac{\mu_m(k)}{2} + \frac{k^2}{2\mu_m(k)} = 0. \] (13)
Thus we have that
\[ H_3 = \int d^d k \mu_m(k)a^*(k)a(k). \] (14)
Now we want to show that \( d\Gamma(\mu_m(p)) = \int d^d k \mu_m(k)a^*(k)a(k) \). Let \( \psi_i \in D, \psi_i = \{ \psi_i^{(n)} \}_{n \in \mathbb{N}} \). We have
\[ \langle \psi_2, d\Gamma(\mu_m(p))\psi_1 \rangle = \sum_{n=1}^{\infty} \int d^d k \langle \psi_2^{(n)}(k_1, \ldots, k_n)(\mu_m(k_1) + \cdots + \mu_m(k_n))\psi_1^{(n)}(k_1, \ldots, k_n) \rangle \] (15)
On the other hand

\[
(a(k)\psi_i)^{(n)}(k_1, \ldots, k_n) = \sqrt{n + 1}\psi_i^{(n+1)}(k, k_1, \ldots, k_n).
\]

Hence

\[
\int d^d k \mu_m(k) (a(k)\psi_1, a(k)\psi_2)
= \sum_{n=0}^{\infty} (n+1) \int d^d k \mu_m(k) \int d^{nd} k \overline{\psi_1^{(n+1)}}(k, k_1, \ldots, k_n) \psi_2^{(n+1)}(k, k_1, \ldots, k_n)
= \sum_{n=0}^{\infty} \int d^{(n+1)d} k (\mu_m(k_1) + \cdots + \mu_m(k_{n+1})) \overline{\psi_1^{(n+1)}}(k_1, \ldots, k_{n+1}) \psi_2^{(n+1)}(k_1, \ldots, k_{n+1})
\]

(17)

**Problem 2.** The interaction Hamiltonian

\[
H_I(g) := \lambda \int d^d x g(x) : \varphi(x)^4 : , \quad g \in C_0^\infty(\mathbb{R}^d), \quad \lambda > 0,
\]

is well defined as a quadratic form on \( D \times D \) (this can be taken for granted). Show that this quadratic form cannot arise from an operator containing \( \Omega \) in its domain for \( d > 1 \). Hint: Consider the formal expression for \( H_I(g)\Omega \) which is of the form \((0, 0, 0, \psi^{(4)}, 0, \ldots)\) and show that \( \psi^{(4)} \) is not square integrable.

**Solution:** We compute

\[
H_I(g)\Omega
= \lambda \int d^d x g(x)(2\pi)^{-2d} \int \frac{d^d k_1}{\sqrt{2\mu_m(k_1)}} \cdots \frac{d^d k_4}{\sqrt{2\mu_m(k_4)}} e^{-i(k_1 + \cdots + k_4)x} a^*(k_1) \cdots a^*(k_4)\Omega
= \lambda (2\pi)^{-2d+d/2} \int \frac{d^d k_1}{\sqrt{2\mu_m(k_1)}} \cdots \frac{d^d k_4}{\sqrt{2\mu_m(k_4)}} \hat{g}(k_1 + \cdots + k_4) a^*(k_1) \cdots a^*(k_4)\Omega
\]

(19)

From this we read off that

\[
\psi^{(4)}(k_1, \ldots, k_4) = \lambda (2\pi)^{-2d+d/2} \sqrt{4!} \frac{1}{\sqrt{2\mu_m(k_1)}} \cdots \frac{1}{\sqrt{2\mu_m(k_4)}} \hat{g}(k_1 + \cdots + k_4)
\]

(20)

We want to show that the following integral diverges:

\[
I := \int \frac{d^d k_1}{\mu_m(k_1)} \cdots \frac{d^d k_4}{\mu_m(k_4)} |\hat{g}(k_1 + \cdots + k_4)|^2
\]

(21)

We set \( k := k_1 + \cdots + k_4 \). Then

\[
I = \int d^d k d^d k_2 d^d k_3 \frac{|\hat{g}(k)|^2}{\mu_m(k_2)\mu_m(k_3)} \int d^d k_4 \frac{1}{\mu_m(k-k_2-k_3-k_4)} \frac{1}{\mu_m(k_4)}
\]

(22)
Let us denote \( K = k - k_2 - k_3 \). It suffices to show that the following integral diverges for any \( K \):

\[
I_K := \int d^d k_4 \frac{1}{\mu_m(K - k_4)} \frac{1}{\mu_m(k_4)} \geq \int_{|k_4| \geq 1} d^d k_4 \frac{1}{\mu_m(K - k_4)} \frac{1}{\mu_m(k_4)}
\]

\[
\geq C_K \int_{|k_4| \geq 1} \frac{d^d k_4}{|k_4|^2},
\]

where \( C_K > 0 \). Since \( d > 1 \) this integral diverges.

**Problem 3.** Let \( \mathcal{D} = S(\mathbb{R}^d), d = 3 \), be the symplectic space with the standard symplectic form. Consider the representations of \( \mathcal{W} \) on Fock space given by

\[
\rho_{\mu_m}(W(f)) = e^{i(\varphi_{\mu_m}(\text{Re} f) + \pi_{\mu_m}(\text{Im} f))}.
\]

These representations are irreducible (this can be taken for granted). Show that \( \rho_{\mu_{m_1}} \) is not unitarily equivalent to \( \rho_{\mu_{m_2}} \) if \( m_1 \neq m_2, m_1, m_2 > 0 \). Hints:

(i) Suppose, by contradiction, that there is a unitary \( T \) on Fock space which intertwines the two representations. Let \( E \ni (a, R) \mapsto U_{\mu}(a, R) \) be the unitary representation of the group of Euclidean motions in the \( t = 0 \) plane (space translations and rotations) which implements the corresponding automorphisms in the two representations. Show that \( C(a, R) := T^{-1} U(a, R)^* T U(a, R) \) must be a multiple of the identity.

(ii) Use that \( E \) has no non-trivial one-dimensional representations.

(iii) Use that multiples of \( \Omega \) are the only vectors in Fock space invariant under \( (a, R) \mapsto U(a, R) \).

**Solution:** Suppose there is a unitary \( T \) on Fock space s.t. for all \( f \in S(\mathbb{R}^d) \)

\[
T \rho_1(W(f)) T^{-1} = \rho_2(W(f)),
\]

where we set \( \rho_i := \rho_{\mu_{m_i}} \).

Recall that for any mass \( m \) we have a unitary representation of the Poincare group \( \mathcal{P}_+ \ni (\widetilde{x}, \Lambda) \mapsto U_m(\widetilde{x}, \Lambda) \) acting on the Fock space. Consider a subgroup \( E \subset \mathcal{P}_+ \) of Euclidean motions on the \( t = 0 \) plane (i.e. space-translations and rotations). For \( (a, R) \in E \) the representation

\[
U_m(a, R) = \Gamma(u_{(a,R)}), \quad \hat{u}_{(a,R)}\hat{g}(p) = e^{-i p a} \hat{g}(R^{-1} p), \text{ or } (u_{(a,R)} g)(x) = g(R^{-1}(x - a)),
\]

(where \( g \in L^2(\mathbb{R}^d) \)), is in fact independent of \( m \) so we can drop the subscript. We have

\[
U(a, R)\rho_1(W(f))U(a, R)^* = \rho_1(W(S_{(a,R)} f)),
\]

\[
U(a, R)\rho_2(W(f))U(a, R)^* = \rho_2(W(S_{(a,R)} f)),
\]

where
where \((S_{(a,R)}f)(x) = f(R^{-1}(x-a))\). Thus we get

\[
U(a,R)\rho_1(W(f))U(a,R)^* = \rho_1(W(S_{(a,R)}f)) = T^{-1}\rho_2(W(S_{(a,R)}f))T
\]

\[
= T^{-1}U(a,R)\rho_2(W(S_{(a,R)}f))U(a,R)^*T = T^{-1}U(a,R)T\rho_1(W(f))T^{-1}U(a,R)^*T
\]

Hence \(C(a,R) := T^{-1}U(a,R)^*TU(a,R)\) commutes with all \(\rho_1(W(f))\) and thus, by irreducibility, is a multiple of identity s.t. \(|C(a,R)| = 1\). Thus we have

\[
TU(a,R)T^{-1} = U(a,R)C(a,R).
\]

(30)

It easily follows from this relation that \(C(a,R)\) is a one-dimensional representation of \(E\) and thus identity representation: \(C(a,R) = 1\). Consequently,

\[
TU(a,R) = U(a,R)T, \text{ and hence } T\Omega = U(a,R)T\Omega.
\]

(31)

Since \(\Omega\) is the only (up to a multiple) vector in Fock space invariant under \(U(a,R)\), we have that \(T\Omega = c\Omega, |c| = 1\). Therefore

\[
\langle \Omega, \rho_1(W(f))\Omega \rangle = \langle \Omega, \rho_2(W(f))\Omega \rangle, \text{ hence } e^{-\frac{1}{2}||f_{\mu m_1}||^2} = e^{-\frac{1}{2}||f_{\mu m_2}||^2},
\]

(32)

and consequently \(||f_{\mu m_1}||^2 = ||f_{\mu m_2}||^2\). This is a contradiction, because e.g. for \(f\) s.t. \(\text{Im } f = 0,\)

\[
(m_1^2 - m_2^2) \int \frac{d^dk}{\mu_{m_1}(k)\mu_{m_2}(k)(\mu_{m_1}(k) + \mu_{m_2}(k))} |\hat{f}(k)|^2 = 0.
\]

(33)

**Additional Problem:** Let \(E \ni (a,R) \mapsto U(a,R)\) be a unitary representation of the group of Euclidean motions in \(\mathbb{R}^3\) in a one-dimensional Hilbert space. Show that this representation is trivial.

**Solution:** By the Stone’s theorem, we have

\[
U(a,1) = e^{-ipa}
\]

(34)

for some \(p \in \mathbb{R}^3\). The multiplication law gives

\[
U(0,R)U(a,1)U(0,R^{-1}) = U(Ra,R)U(0,R^{-1}) = U(Ra,1) = e^{-ip(Ra)} = e^{-i(R^{-1}p)a}
\]

(35)

On the other hand, since the representation is one-dimensional:

\[
U(0,R)U(a,1)U(0,R^{-1}) = U(a,1) = e^{-ipa}.
\]

(36)

By differentiating w.r.t. \(a\) at zero, we have

\[
R^{-1}p = p
\]

(37)

for all rotations \(R\) so, \(p = 0\). Thus \(U\) is at best a non-trivial representation of \(SO(3)\). But we know all irreducible finite-dimensional representations of \(SO(3)\) (recall from Quantum Mechanics, angular momentum) and it has no non-trivial irreducible representations.
**Additional Problem:** Show that $\Omega$ is the only vector in the Fock space invariant under space translations.

**Solution:** Let $\psi = \{\psi^{(n)}\}_{n \in \mathbb{N}}$ be a unit vector orthogonal to $\Omega$ which is invariant under space translations. Then

$$1 = \langle \psi, \psi \rangle = \langle \psi, U(a, 1)\psi \rangle = \sum_{n \geq 1} \int d^{nd}k |\psi|^2(k_1, \ldots, k_n)e^{-ia(k_1+\cdots+k_n)} \to 0$$

as $a \to \infty$ by the Riemann-Lebesgue lemma. This is a contradiction.

**To be discussed in class:** 29.06.2017