Dr. Wojciech Dybalski
Dr. Daniela Cadamuro

Technische Universität München
Sommersemester 2017

Algebraic Quantum Field Theory
Homework Sheet 4

Problem 1. Show that in the sense of quadratic forms on $D \times D$, where

$$D = \{ \Psi \in \Gamma_{\rm fin}(\mathfrak{h}) \,|\, \Psi^{(n)} \in S(\mathbb{R}^{nd}) \text{ for all } n \}, \tag{1}$$

we have the following representations for the (free) Hamiltonian $H := \mathrm{d}\Gamma(\mu_m)$:

$$\begin{aligned} \mathrm{d}\Gamma(\mu_m(p)) &= \int d^dk\, \mu_m(k) a^*(k) a(k) \\ &= \frac{1}{2} \int d^dx \big(:\pi^2_{\mu_m}(x): + :\nabla\varphi^2_{\mu_m}(x): + m^2 :\varphi^2_{\mu_m}(x): \big). \end{aligned} \tag{2}$$

Here $\mu_m(p) = \sqrt{p^2 + m^2}$ and

$$\varphi_{\mu_m}(x) := \frac{1}{(2\pi)^{d/2}} \int \frac{d^dk}{\sqrt{2\mu_m(k)}} \big(e^{-ikx} a^*(k) + e^{ikx} a(k) \big), \tag{3}$$

$$\pi_{\mu_m}(x) := \frac{i}{(2\pi)^{d/2}} \int d^dk \sqrt{\frac{\mu_m(k)}{2}} \big(e^{-ikx} a^*(k) - e^{ikx} a(k) \big). \tag{4}$$

The Wick ordering $:(\cdots):$ means shifting creation operators to the left and annihilation operators to the right, ignoring the commutators. For example

$$: (a^*(k_1)a^*(k_2) + a^*(k_1)a(k_2) + a(k_1)a^*(k_2) + a(k_1)a(k_2)) : \tag{5}$$

$$= a^*(k_1)a^*(k_2) + a^*(k_1)a(k_2) + a^*(k_2)a(k_1) + a(k_1)a(k_2). \tag{6}$$

Solution: Define

$$\phi_h(x) := \frac{1}{(2\pi)^{d/2}} \int d^dk (h(k) e^{-ikx} a^*(k) + \overline{h(k)} e^{ikx} a(k)). \tag{7}$$

For $h_1(k) := \frac{m}{\sqrt{2\mu_m(k)}}$ we get $m\varphi(x)$, for $h_2(k) := i\sqrt{\frac{\mu_m(k)}{2}}$ we get $\pi(x)$ for $h_3(k) := -i\frac{k}{\sqrt{2\mu_m(k)}}$ we get $\nabla\varphi(x)$. We consider matrix elements $\langle \psi_1, :\phi_h(x)^2: \psi_2 \rangle$, where $\psi_1, \psi_2 \in D$. This gives rise to expressions $\langle \psi_1, : a^*(k_1)a(k_2) : \psi_2 \rangle$ etc. By definition of $a(k)$ and D these expressions are Schwartz class functions of k_1, k_2. This observation justifies the manipulations below. We do not write ψ_1, ψ_2 explicitly, but they are always understood. We compute:

$$\int d^dx : \phi_h(x)^2 := \frac{1}{(2\pi)^d} \int d^dx \int d^dk_1 d^dk_2 : (h(k_1)e^{-ik_1x}a^*(k_1) + \overline{h(k_1)}e^{ik_1x}a(k_1)) \tag{8}$$

$$\times (h(k_2)e^{-ik_2x}a^*(k_2) + \overline{h(k_2)}e^{ik_2x}a(k_2)) : \tag{9}$$

This gives

$$\begin{aligned} \int d^dx : \phi_h(x)^2 : \; = \int d^dk_1 d^dk_2 \Big(& a^*(k_1)a^*(k_2)h(k_1)h(k_2)\delta(k_1+k_2) \\ & + a^*(k_1)a(k_2)h(k_1)\overline{h(k_2)}\delta(k_1-k_2) \\ & + a^*(k_2)a(k_1)\overline{h(k_1)}h(k_2)\delta(k_1-k_2) \\ & + a(k_1)a(k_2)\overline{h(k_1)h(k_2)}\delta(k_1+k_2)\Big). \end{aligned} \tag{10}$$

Consequently:

$$\begin{aligned} & \int d^dx : \phi_h(x)^2 : \\ & = \int d^dk \left(a^*(k)a^*(-k)h(k)h(-k) + 2a^*(k)a(k)|h(k)|^2 + a(k)a(-k)\overline{h(k)h(-k)}) \right) \\ & = \int d^dk \, 2a^*(k)a(k)|h(k)|^2 + \int d^dk \Big(a^*(k)a^*(-k)h(k)h(-k) + \text{h.c.} \Big) \end{aligned} \tag{11}$$

The last expression on the r.h.s. of (2), let's call it H_3, is given by

$$H_3 = \frac{1}{2} \sum_{i=1}^{3} \int d^dx : \phi_{h_i}(x)^2 : \tag{12}$$

But we have

$$\begin{aligned} 2\sum_{i=1}^{3} |h_i(k)|^2 &= 2\Big(\frac{m^2}{2\mu_m(k)} + \frac{\mu_m(k)}{2} + \frac{k^2}{2\mu_m(k)}\Big) = \mu_m(k), \\ \sum_{i=1}^{3} h_i(k)h_i(-k) &= \frac{m^2}{2\mu_m(k)} - \frac{\mu_m(k)}{2} + \frac{k^2}{2\mu_m(k)} = 0. \end{aligned} \tag{13}$$

Thus we have that

$$H_3 = \int d^dk \, \mu_m(k)a^*(k)a(k). \tag{14}$$

Now we want to show that $\mathrm{d}\Gamma(\mu_m(p)) = \int d^dk \, \mu_m(k)a^*(k)a(k)$. Let $\psi_i \in D$, $\psi_i = \{\psi_i^{(n)}\}_{n\in\mathbb{N}}$. We have

$$\langle \psi_2, \mathrm{d}\Gamma(\mu_m(p))\psi_1 \rangle = \sum_{n=1}^{\infty} \int d^{nd}k \; \overline{\psi_2}^{(n)}(k_1,\ldots,k_n)(\mu_m(k_1)+\cdots+\mu_m(k_n))\psi_1^{(n)}(k_1,\ldots,k_n). \tag{15}$$

On the other hand

$$(a(k)\psi_i)^{(n)}(k_1,\dots,k_n) = \sqrt{n+1}\psi_i^{(n+1)}(k,k_1,\dots,k_n). \tag{16}$$

Hence

$$\begin{aligned}
&\int d^dk\, \mu_m(k)\langle a(k)\psi_1, a(k)\psi_2\rangle \\
&= \sum_{n=0}^{\infty}(n+1)\int d^dk\, \mu_m(k)\int d^{nd}k\, \overline{\psi_1}^{(n+1)}(k,k_1,\dots,k_n)\psi_2^{(n+1)}(k,k_1,\dots,k_n) \\
&= \sum_{n=0}^{\infty}\int d^{(n+1)d}k\, (\mu_m(k_1)+\dots+\mu_m(k_{n+1}))\overline{\psi_1}^{(n+1)}(k_1,\dots,k_{n+1})\psi_2^{(n+1)}(k_1,\dots,k_{n+1})
\end{aligned} \tag{17}$$

Problem 2. The interaction Hamiltonian

$$H_I(g) := \lambda\int d^dx\, g(x) : \varphi(x)^4 :, \quad g\in C_0^\infty(\mathbb{R}^d), \quad \lambda>0, \tag{18}$$

is well defined as a quadratic form on $D\times D$ (this can be taken for granted). Show that this quadratic form cannot arise from an operator containing Ω in its domain for $d>1$. Hint: Consider the formal expression for $H_I(g)\Omega$ which is of the form $(0,0,0,0,\psi^{(4)},0,\dots)$ and show that $\psi^{(4)}$ is not square integrable.

Solution: We compute

$$\begin{aligned}
&H_I(g)\Omega \\
&= \lambda\int d^dx\, g(x)(2\pi)^{-2d}\int \frac{d^dk_1}{\sqrt{2\mu_m(k_1)}}\cdots\frac{d^dk_4}{\sqrt{2\mu_m(k_4)}}e^{-i(k_1+\dots+k_4)x}a^*(k_1)\dots a^*(k_4)\Omega \\
&= \lambda(2\pi)^{-2d+d/2}\int \frac{d^dk_1}{\sqrt{2\mu_m(k_1)}}\cdots\frac{d^dk_4}{\sqrt{2\mu_m(k_4)}}\widehat{g}(k_1+\dots+k_4)a^*(k_1)\dots a^*(k_4)\Omega
\end{aligned} \tag{19}$$

From this we read off that

$$\psi^{(4)}(k_1,\dots,k_4) = \lambda(2\pi)^{-2d+d/2}\sqrt{4!}\frac{1}{\sqrt{2\mu_m(k_1)}}\cdots\frac{1}{\sqrt{2\mu_m(k_4)}}\widehat{g}(k_1+\dots+k_4) \tag{20}$$

We want to show that the following integral diverges:

$$I := \int \frac{d^dk_1}{\mu_m(k_1)}\cdots\frac{d^dk_4}{\mu_m(k_4)}|\widehat{g}(k_1+\dots+k_4)|^2 \tag{21}$$

We set $k := k_1+\dots+k_4$. Then

$$I = \int d^dk d^dk_2 d^dk_3 \frac{|\widehat{g}(k)|^2}{\mu_m(k_2)\mu_m(k_3)}\int d^dk_4 \frac{1}{\mu_m(k-k_2-k_3-k_4)}\frac{1}{\mu_m(k_4)} \tag{22}$$

Let us denote $K = k - k_2 - k_3$. It suffices to show that the following integral diverges for any K:

$$\begin{aligned} I_K &:= \int d^d k_4 \frac{1}{\mu_m(K-k_4)} \frac{1}{\mu_m(k_4)} \geq \int_{|k_4|\geq 1} d^d k_4 \frac{1}{\mu_m(K-k_4)} \frac{1}{\mu_m(k_4)} \\ &\geq C_K \int_{|k_4|\geq 1} \frac{d^d k_4}{|k_4|^2}, \end{aligned} \tag{23}$$

where $C_K > 0$. Since $d > 1$ this integral diverges.

Problem 3. Let $\mathcal{D} = S(\mathbb{R}^d)$, $d = 3$, be the symplectic space with the standard symplectic form. Consider the representations of $\mathcal{W}$ on Fock space given by

$$\rho_{\mu_m}(W(f)) = e^{i(\varphi_{\mu_m}(\mathrm{Re}\, f) + \pi_{\mu_m}(\mathrm{Im}\, f))}. \tag{24}$$

These representations are irreducible (this can be taken for granted). Show that $\rho_{\mu_{m_1}}$ is not unitarily equivalent to $\rho_{\mu_{m_2}}$ if $m_1 \neq m_2$, $m_1, m_2 > 0$. Hints:

(i) Suppose, by contradiction, that there is a unitary T on Fock space which intertwines the two representations. Let $E \ni (a, R) \to U(a, R)$ be the unitary representation of the group of Euclidean motions in the $t = 0$ plane (space translations and rotations) which implements the corresponding automorphisms in the two representations. Show that $C(a, R) := T^{-1}U(a, R)^* T U(a, R)$ must be a multiple of the identity.

(ii) Use that E has no non-trivial one-dimensional representations.

(iii) Use that multiples of Ω are the only vectors in Fock space invariant under $(a, R) \to U(a, R)$.

Solution: Suppose there is a unitary T on Fock space s.t. for all $f \in S(\mathbb{R}^d)$

$$T\rho_1(W(f))T^{-1} = \rho_2(W(f)), \tag{25}$$

where we set $\rho_i := \rho_{\mu_{m_i}}$.

Recall that for any mass m we have a unitary representation of the Poincare group $P_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto U_m(\tilde{x}, \Lambda)$ acting on the Fock space. Consider a subgroup $E \subset P_+^\uparrow$ of Euclidean motions on the $t = 0$ plane (i.e. space-translations and rotations). For $(a, R) \in E$ the representation

$$U_m(a, R) = \Gamma(u_{(a,R)}), \quad \widehat{u_{(a,R)}g}(p) = e^{-ipa}\widehat{g}(R^{-1}p), \text{ or } (u_{(a,R)}g)(x) = g(R^{-1}(x - a)), \tag{26}$$

(where $g \in L^2(\mathbb{R}^d)$), is in fact independent of m so we can drop the subscript. We have

$$U(a, R)\rho_1(W(f))U(a, R)^* = \rho_1(W(S_{(a,R)}f)), \tag{27}$$

$$U(a, R)\rho_2(W(f))U(a, R)^* = \rho_2(W(S_{(a,R)}f)), \tag{28}$$

where $(S_{(a,R)}f)(x) = f(R^{-1}(x-a))$. Thus we get

$$\begin{aligned} &U(a,R)\rho_1(W(f))U(a,R)^* = \rho_1(W(S_{(a,R)}f)) = T^{-1}\rho_2(W(S_{(a,R)}f))T \\ &= T^{-1}U(a,R)\rho_2(W(S_{(a,R)}f))U(a,R)^*T = T^{-1}U(a,R)T\rho_1(W(f))T^{-1}U(a,R)^*T. \end{aligned} \tag{29}$$

Hence $C(a,R) := T^{-1}U(a,R)^*TU(a,R)$ commutes with all $\rho_1(W(f))$ and thus, by irreducibility, is a multiple of identity s.t. $|C(a,R)| = 1$. Thus we have

$$TU(a,R)T^{-1} = U(a,R)C(a,R). \tag{30}$$

It easily follows from this relation that $C(a,R)$ is a one-dimensional representation of E and thus identity representation: $C(a,R) = 1$. Consequently,

$$TU(a,R) = U(a,R)T, \text{ and hence } T\Omega = U(a,R)T\Omega. \tag{31}$$

Since Ω is the only (up to a multiple) vector in Fock space invariant under $U(a,R)$, we have that $T\Omega = c\Omega$, $|c| = 1$. Therefore

$$\langle\Omega, \rho_1(W(f))\Omega\rangle = \langle\Omega, \rho_2(W(f))\Omega\rangle, \text{ hence } e^{-\frac{1}{2}\|f_{\mu_{m_1}}\|^2} = e^{-\frac{1}{2}\|f_{\mu_{m_2}}\|^2}, \tag{32}$$

and consequently $\|f_{\mu_{m_1}}\|^2 = \|f_{\mu_{m_2}}\|^2$. This is a contradiction, because e.g. for f s.t. $\mathrm{Im}\, f = 0$,

$$(m_1^2 - m_2^2)\int \frac{d^dk}{\mu_{m_1}(k)\mu_{m_2}(k)(\mu_{m_1}(k) + \mu_{m_2}(k))}|\widehat{f}(k)|^2 = 0. \tag{33}$$

Additional Problem: Let $E \ni (a,R) \mapsto U(a,R)$ be a unitary representation of the group of Euclidean motions in $\mathbb{R}^3$ in a one-dimensional Hilbert space. Show that this representation is trivial.

Solution: By the Stone's theorem, we have

$$U(a,1) = e^{-ipa} \tag{34}$$

for some $p \in \mathbb{R}^3$. The multiplication law gives

$$U(0,R)U(a,1)U(0,R^{-1}) = U(Ra,R)U(0,R^{-1}) = U(Ra,1) = e^{-ip(Ra)} = e^{-i(R^{-1}p)a} \tag{35}$$

On the other hand, since the representation is one-dimensional:

$$U(0,R)U(a,1)U(0,R^{-1}) = U(a,1) = e^{-ipa}. \tag{36}$$

By differentiating w.r.t. a at zero, we have

$$R^{-1}p = p \tag{37}$$

for all rotations R so, $p = 0$. Thus U is at best a non-trivial representation of $SO(3)$. But we know all irreducible finite-dimensional representations of $SO(3)$ (recall from Quantum Mechanics, angular momentum) and it has no non-trivial irreducible representations.

Additional Problem: Show that Ω is the only vector in the Fock space invariant under space translations.

Solution: Let $\psi = \{\psi^{(n)}\}_{n\in\mathbb{N}}$ be a unit vector orthogonal to Ω which is invariant undar space translations. Then

$$1 = \langle \psi, \psi \rangle = \langle \psi, U(a,1)\psi \rangle = \sum_{n\geq 1} \int d^{nd}k \, |\psi|^2(k_1, \dots, k_n) e^{-ia(k_1+\cdots+k_n)} \to 0 \qquad (38)$$

as $a \to \infty$ by the Riemann-Lebesgue lemma. This is a contradiction.

To be discussed in class: 29.06.2017