Prof. Dr. Wojciech Dybalski
Maximilian Duell

LMU and TU München
Sommersemester 2017

Algebraic Quantum Field Theory
Homework Sheet 1 - solutions

Problem 1. Let $\mathcal{H}_1 = L^2(\mathbb{R}^n)$ with scalar products $\langle f, g\rangle = \int d^n x\, \overline{f}(x) g(x)$. Show that the prescription

$$\big(\pi_1(W(z))f\big)(x) = e^{\frac{i}{2}uv} e^{ivx} f(x+u), \quad z = u + iv, \tag{1}$$

defines a representation $\mathcal{W}$.

Solution: We extend π by linearity to arbitrary elements of $\mathcal{W}$:

$$\pi_1(\sum_i c_i W(z_i)) := \sum_i c_i \pi_1(W(z_i)) \tag{2}$$

so linearity holds by construction. Now multiplicativity: Let us first show that

$$\pi_1(W(z)W(z')) = \pi_1(W(z))\pi_1(W(z')). \tag{3}$$

We compute the l.h.s. of (3) of some $f \in L^2(\mathbb{R}^n)$:

$$\begin{aligned}\big(\pi_1(W(z)W(z'))f\big)(x) &= e^{\frac{i}{2}\mathrm{Im}\langle z,z'\rangle}\big(\pi_1(W(z+z'))f\big)(x) \\ &= e^{\frac{i}{2}\mathrm{Im}\langle z,z'\rangle} e^{\frac{i}{2}(u+u')(v+v')} e^{i(v+v')x} f(x+u+u').\end{aligned} \tag{4}$$

Now we compute the r.h.s. of (3):

$$\begin{aligned}(\pi_1(W(z))\pi_1(W(z'))f)(x) &= e^{\frac{i}{2}uv} e^{ivx} (\pi_1(W(z'))f)(x+u) \\ &= e^{\frac{i}{2}uv} e^{ivx} e^{\frac{i}{2}u'v'} e^{iv'(x+u)} f(x+u+u') \\ &= e^{\frac{i}{2}uv} e^{\frac{i}{2}u'v'} e^{iv'u} e^{i(v+v')x} f(x+u+u')\end{aligned} \tag{5}$$

Since $\mathrm{Im}\langle z, z'\rangle = -vu' + uv'$, we obtain that (4) coincides with (5) thus we have (3). Now for general elements of $\mathcal{W}$.

$$\begin{aligned}\pi_1\Big(\big(\sum_i c_i W(z_i)\big)\big(\sum_j c'_j W(z'_j)\big)\Big) &= \sum_{i,j} c_i c'_j \pi_1(W(z_i)W(z'_j)) \\ &= \sum_{i,j} c_i c'_j \pi_1(W(z_i))\pi_1(W(z'_j)) \\ &= \pi_1\big(\sum_i c_i W(z_i)\big)\pi_1\big(\sum_j c'_j W(z'_j)\big).\end{aligned} \tag{6}$$

Finally, we show that $\pi_1(W^*) = \pi_1(W)^*$. Clearly, it is enough to show this on a Weyl operator:

$$\langle g, \pi_1(W(z)^*)f\rangle = \int d^n x\, \overline{g(x)}\big(\pi_1(W(-z))f\big)(x) = \int d^n x\, \overline{g(x)} e^{\frac{i}{2}uv} e^{-ivx} f(x-u). \tag{7}$$

On the other hand

$$\begin{aligned}\langle g, \pi_1(W(z))^* f\rangle &= \langle \pi_1(W(z))g, f\rangle = \int d^n x\, e^{-\frac{i}{2}uv} e^{-ivx} \overline{g(x+u)} f(x) \\ &= \int d^n x\, e^{\frac{i}{2}uv} e^{-ivx} \overline{g(x)} f(x-u).\end{aligned} \tag{8}$$

Problem 2. Let $\mathcal{H}_2 = L^2(\mathbb{R}^n)$ with scalar products $\langle f, g\rangle = \int d^n x\, \overline{f}(x) g(x)$. One defines

$$\big(\pi_2(W(z))f\big)(x) = e^{-\frac{i}{2}uv} e^{iux} f(x-v), \quad z = u + iv. \tag{9}$$

Show that this prescription defines a representation of $\mathcal{W}$. Also show that it is unitarily equivalent to the representation from Problem 1 with the Fourier transform being the unitary.

Solution. Since we know that π_1 is a representation, it is enough to check the relation

$$\big((\mathcal{F}\pi_1(W)\mathcal{F}^{-1})f\big)(x) = (\pi_2(W)f)(x). \tag{10}$$

By linearity, it is enough to show this for $W = W(z)$ and $f \in S(\mathbb{R}^n)$. We have

$$\begin{aligned}\big((\mathcal{F}\pi_1(W(z))\mathcal{F}^{-1})f\big)(x) &= \frac{1}{(2\pi)^{n/2}} \int d^n y\, e^{-ixy} (\pi_1(W(z))\mathcal{F}^{-1}f)(y) \\ &= \frac{1}{(2\pi)^{n/2}} \int d^n y\, e^{-ixy} e^{\frac{i}{2}uv} e^{ivy} (\mathcal{F}^{-1}f)(y+u) \\ &= \frac{1}{(2\pi)^{n/2}} \int d^n y\, e^{-ix(y-u)} e^{\frac{i}{2}uv} e^{iv(y-u)} (\mathcal{F}^{-1}f)(y) \\ &= e^{ixu} e^{-\frac{i}{2}uv} \frac{1}{(2\pi)^{n/2}} \int d^n y\, e^{-i(x-v)y} (\mathcal{F}^{-1}f)(y) \\ &= e^{-\frac{i}{2}uv} e^{ixu} f(x-v).\end{aligned} \tag{11}$$

Problem 3. Show that there is no representation of $\mathcal{W}$ on a finite dimensional Hilbert space (apart from the trivial one $\pi(W) = 0$ for all $W \in \mathcal{W}$). Hints:

(i) First check that you can assume without loss of generality that $\pi(1) = 1$.

(ii) Next use properties of the determinant and the relation

$$W(z)W(z')W(z)^* = e^{i\mathrm{Im}\langle z, z'\rangle} W(z'). \tag{12}$$

Solution. First, let π' be a representation of $\mathcal{W}$ on a finite dimensional Hilbert space $\mathcal{H}'$ s.t. $\pi'(1) \neq 1$. We know that $\pi'(1)$ is a projection and it cannot be 0 since then the representation would be trivial. Note that $\mathcal{H} := \operatorname{Ran} \pi'(1)$ is an invariant subspace of $\mathcal{H}'$ (that is $\pi'(W)\mathcal{H} \subset \mathcal{H}$ for all $W \in \mathcal{W}$) since $\pi'(1)$ commutes with all $\pi(W)$). Thus we can restrict π' to $\mathcal{H}$ and call this restricted representation π. Clearly $\pi(1) = 1$.

Note that $\pi(W(z))\pi(W(z))^* = \pi(W(z))\pi(W(-z)) = \pi(W(z)W(-z)) = \pi(W(0)) = 1$ i.e. $\pi(W(z))$ are unitary. Consequently

$$1 = \det(\pi(W(z))\pi(W(z))^*) = |\det(\pi(W(z))|^2, \tag{13}$$

hence $\det(\pi(W(z))) = e^{i\phi(z)}$, in particular $\det(\pi(W(z))) \neq 0$. Next we write using invariance of the determinant under conjugation of the matrix with a unitary:

$$\det(\pi(W(z'))) = \det(\pi(W(z))\pi(W(z'))\pi(W(z))^*) = e^{i\mathrm{Im}\langle z,z'\rangle d}\det(\pi(W(z'))), \tag{14}$$

where d is the dimension of $\mathcal{H}$. Thus we have

$$1 = e^{i\mathrm{Im}\langle z,z'\rangle d} \tag{15}$$

for all z, z' which is a contradiction.

Problem 4. Show that the representation from Problem 1 is irreducible. Hints:

(i) It suffices to show that given $f, g \in L^2(\mathbb{R}^n)$, $f \neq 0$, the equality

$$\langle g, \pi_1(W(z))f\rangle = 0 \tag{16}$$

for all z implies $g = 0$. (This implies that f is cyclic for $\pi_1(\mathcal{W})$).

(ii) Use that the Fourier transform is injective on $L^1(\mathbb{R}^n)$.

Solution.

We rewrite the above condition as follows

$$e^{\frac{i}{2}uv}\int \overline{g(x)}e^{ivx}f(x+u)dx = 0 \tag{17}$$

That is

$$\int e^{ivx}\overline{g(x)}f(x+u)dx = 0. \tag{18}$$

Since the Fourier transform is injective on $L^1(\mathbb{R}^n)$, we have that

$$\overline{g(x)}f(x+u) = 0 \tag{19}$$

for any u as an element of $L^1(\mathbb{R}^n)$. This is an L^2 function in u, thus I can write

$$0 = \int du\, |\overline{g(x)}f(x+u)|^2 = |g(x)|^2\|f\|_2^2. \tag{20}$$

Now we integrate over x to get

$$0 = \|g\|_2^2\|f\|_2^2. \tag{21}$$

Since $\|f\|_2 \neq 0$ we have $g = 0$ as an element of $L^2(\mathbb{R}^n)$.