The "magic formula" for linearly edge-reinforced random walks

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Abstract

Linearly edge-reinforced random walk on a finite graph is a mixture of reversible Markov chains with an explicitly known mixing measure. We give a new proof of this fact.

Key Words and Phrases: Linearly edge-reinforced random walk, mixing measure

1 Introduction

10 years ago, when the first PhD student (S.R.) started to work at the newly opened institute Eurandom, Mike Keane introduced her to linearly edge-reinforced random walk. A few years before, he had discovered a formula for the asymptotic behaviour of this stochastic process on a triangle; see KEANE (1990). Already in 1986, Coppersmith and Diaconis had discovered a description of the asymptotic behaviour of linearly edge-reinforced random walk on a general finite graph. This result was announced in DIACONIS (1988); however, its proof was only written down in an unpublished manuscript COPPERSMITH and DIACONIS (1986), which Mike Keane did not have. He thought that understanding this "magic formula" was important to understand, among others, recurrence properties of the reinforced random walk. Hence, Mike Keane and S.R. developed a proof of this formula; see KEANE and ROLLES (2000). One year after Eurandom had opened, S.R. told a new postdoc of Eurandom (F.M.) about reinforced random walk. Mike Keane's vision about reinforced random walk became reality, culminating in the recurrence proof of reinforced random walks on certain two-dimensional graphs; see MERKL and ROLLES (2007). An essential ingredient of all recurrence proofs is the "magic formula". The "magic formula" originally describes the asymptotics of local times spent on the edges.

However, there is another view of this formula: linearly edge-reinforced random walk is a mixture of time-homogeneous Markov chains, and the mixing measure is again given by the "magic formula". It has always been astonishing why an *asymptotic* analysis of the process is needed to describe even a *single* step as a mixture. Given the importance of the "magic formula", we were wondering whether there was a proof of the description of linearly edge-reinforced random walk as a mixture without using asymptotics. In this paper, we provide such a purely calculational proof, which is completely independent of the proof given in KEANE and ROLLES (2000).

1.1 The model

Let G be a connected, finite graph without direct loops, consisting of a vertex set V and an edge set $E \subseteq \{\{u,v\}: \ u,v \in V, \ u \neq v\}$. All relevant information is encoded in the incidence matrix $I: V \times E \to \{0,1\}$, where $I_{ve} = 1$ means that the vertex v is one of the two endpoints of the edge e, i.e. $v \in e$.

Linearly edge-reinforced random walk on G with starting point $v_0 \in V$ and initial weights $a = (a_e)_{e \in E} \in (0,\infty)^E$ is a stochastic process $(X_n)_{n \in \mathbb{N}_0}$ with law P_a determined by the following conditions:

$$P_a[X_0 = v_0] = 1, \tag{1}$$

$$P_a[X_{n+1} = v | X_0, \ldots, X_n] = \frac{w_{\{X_n, v\}}(n)}{\sum_{e \in E} I_{X_n e} w_e(n)} \quad (v \in V, n \in \mathbb{N}_0), \tag{2}$$

$$\text{with} \quad w_e(n) = a_e + \sum_{i=0}^{n-1} 1_{\{X_i, X_{i+1}\} = e} \quad (e \in E, n \in \mathbb{N}_0). \tag{3}$$

We interpret $w_e(n)$ as the weight of edge e at time n. Initially, the edge weights are equal to a. In each time step, the edge-reinforced random walker jumps to a neighboring vertex with probability proportional to the weight of the connecting edge. Each time an edge is traversed, its weight is increased by 1.

Linearly edge-reinforced random walk was introduced by Persi Diaconis in 1986. One can view it as a simple model of a random walker exploring an unknown environment. OTHMER and STEVENS (1997) considered reinforced random walks as a simple model for the motion of myxobacteria; these bacteria produce a slime trail and prefer to glide on the slime produced earlier.

1.2 The "magic formula"

Fix $e_0 \in E$ and abbreviate $e_0^c = E \setminus \{e_0\}$. Let $\Omega_{e_0} = (0,\infty)^{e_0^c}$. For $e \in e_0^c$, let x_e denote the e-th coordinate of $x \in \Omega_{e_0}$. Furthermore, we set $x_{e_0} = 1$. We call x_e the weight of the edge e. For $v \in V$, the weight y_v of the vertex v is defined to be the sum of the weights of edges adjacent to v:

$$y = (y_v)_{v \in V} = Ix, \qquad \text{i.e. } y_v = \sum_{e: v \in e} x_e. \tag{4}$$

Let $\mathcal{T}$ denote the set of all spanning trees of G, where a spanning tree is viewed as a set of edges. We define the following measures on Ω_{e_0}:

$$\mu_{e_0} = \Big(\prod_{v\in V} y_v^{-1/2}\Big)\sqrt{\sum_{T\in\mathcal{T}}\prod_{e\in T} x_e}\prod_{e\in e_0^c}\frac{\mathrm{d}x_e}{x_e}, \qquad \mu_{v_0,e_0} = \sqrt{y_{v_0}}\,\mu_{e_0} \qquad (v_0\in V). \tag{5}$$

For any edge $e=\{u,v\}\in E$, we introduce

$$\xi_e = \frac{x_e}{\sqrt{y_u y_v}}. \tag{6}$$

For given initial weights $a\in(0,\infty)^E$, we abbreviate

$$f_a = \prod_{e\in E}\xi_e^{a_e}. \tag{7}$$

We set

$$z(a,v_0,e_0) = \int_{\Omega_{e_0}} f_a\,\mathrm{d}\mu_{v_0,e_0}. \tag{8}$$

In Lemma 3.5(b) of MERKL and ROLLES (2007), it is shown that $z(a,v_0,e_0)$ does not depend on the choice of the edge e_0; thus we write $z(a,v_0)=z(a,v_0,e_0)$. Furthermore, $0<z(a,v_0)<\infty$; see Lemma 12 below.

THEOREM 1 (MAGIC FORMULA) *Let $\pi=(v_0,\dots,v_n)$ be a finite path in G with starting point v_0 and endpoint v_n. For any edge $e\in E$, let $k_e(\pi)$ denote the number of times the path π traverses the (undirected) edge e. Using the notation $k(\pi)=(k_e(\pi))_{e\in E}$, one has*

$$P_a[X_i=v_i \text{ for all } 0\le i\le n] = \frac{z(a+k(\pi),v_n)}{z(a,v_0)} = \int_{\Omega_{e_0}}\prod_{i=0}^{n-1}\frac{x_{\{v_i,v_{i+1}\}}}{y_{v_i}}\frac{f_a}{z(a,v_0)}\,\mathrm{d}\mu_{v_0,e_0}. \tag{9}$$

Consequently, the edge-reinforced random walk on G is a mixture of reversible Markov chains, and the mixing measure can be described by the measure $f_a\mu_{v_0,e_0}/z(a,v_0)$ on Ω_{e_0}.

The second equation in (9) follows by integrating the following identity with respect to μ_{e_0}:

$$\frac{f_{a+k(\pi)}}{z(a,v_0)}\sqrt{y_{v_n}} = \prod_{i=0}^{n-1}\frac{x_{\{v_i,v_{i+1}\}}}{y_{v_i}}\frac{f_a}{z(a,v_0)}\sqrt{y_{v_0}}. \tag{10}$$

Theorem 1 allows us to use certain Markov chain techniques to prove properties of the non-Markovian edge-reinforced random walk. This strategy has been very fruitful. For instance, it was used in MERKL and ROLLES (2007) to prove bounds for certain hitting probabilities of the edge-reinforced random walk.

1.3 Notation

Throughout the article, we use the following notation and conventions:
Ordering. We order the vertex set V and the edge set E of G in an arbitrary way.
Matrices. For $A \in \mathbb{R}^{J\times K}$, $J' \subseteq J$, and $K' \subseteq K$, we denote by $A_{J'K'} = (A_{jk})_{j\in J',k\in K'}$ the restriction of A to the index set $J' \times K'$. For a set B and $b_1, b_2 \in B$, we set $b_1^c = B \setminus \{b_1\}$ and $(b_1b_2)^c = B \setminus \{b_1, b_2\}$. If $A \in \mathbb{R}^{J\times J}$ is a square matrix, indexed by an ordered index set $J = \{j_1, \dots, j_n\}$, we set, using the notation $\mathrm{sign}_{j_kj_l} = (-1)^{k+l}$:

$$\mathrm{ad}\, A = (\mathrm{sign}_{ij} \det(A_{j^ci^c}))_{i,j\in J}. \tag{11}$$

Note that for invertible matrices A, one has $\mathrm{ad}\, A = (\det A)A^{-1}$ by Cramer's rule. The following formula from linear algebra will be used several times: For any $n \times n$ matrix A and any $n \times n$ matrix Π of rank 1, the following holds:

$$\det(A + \Pi) = \det A + \mathrm{trace}[\mathrm{ad}(A)\Pi]. \tag{12}$$

Incidence matrices. Recall that we denote the unsigned incidence matrix of G by I. The signed incidence matrix $\sigma = (\sigma_{ve})_{v\in V,e\in E}$ is defined as follows: if $e = \{u, v\}$ with $u < v$, then we set $\sigma_{ue} = -1$ and $\sigma_{ve} = 1$. If v is not adjacent to e, then we set $\sigma_{ve} = 0$. We abbreviate $R = I_{Ve_0^c}$ and $R_{v_0} = R_{v_0e_0^c}$, viewed as a row vector.
Special matrices. We introduce the following diagonal matrices: $X = \mathrm{diag}(x_e)_{e\in E}$, $X_{e_0^c} = \mathrm{diag}(x_e)_{e\in e_0^c}$, and $Y = \mathrm{diag}(y_v)_{v\in V}$. We set $W = W^\top = \sigma X\sigma^\top$, $\Pi = \Pi^\top = (I_{ue_0}I_{ve_0})_{u,v\in V} = I_{Ve_0}I_{e_0V}^\top$, and $U = W+\Pi$. Note that the matrix Π has rank 1. There are the following connections between these matrices: First, $W = 2Y - IXI^\top$. To see this identity, let us distinguish two cases: For $v \in V$, the diagonal entry W_{vv} equals $\sum_{e\in E} \sigma_{ve}x_e\sigma_{ve} = \sum_{e\in E} I_{ve}x_e = y_v$, which equals the (v, v)-entry of $2Y - IXI^\top$. For $u, v \in V$, $u \neq v$, the nondiagonal entries W_{uv} and $(2Y - IXI^\top)_{uv}$ both equal $-x_{\{u,v\}}$, if $\{u, v\}$ is an edge in E, and 0 otherwise. Second, we have $IXI^\top = RX_{e_0^c}R^\top + \Pi$. This follows immediately from the definition, using the normalization $x_{e_0} = 1$.
Wedge products. For 1-forms $\omega_1, \dots, \omega_n$, the wedge product $\omega_1 \wedge \dots \wedge \omega_n$ is the n-form defined by $\omega_1 \wedge \dots \wedge \omega_n(\phi_1, \dots, \phi_n) = \det((\omega_i(\phi_j))_{i,j=1,\dots,n})$ for any vectors $\phi_1, \dots, \phi_n$. Note that the wedge product of 1-forms is anticommuting. For background information on the wedge product, see e.g. the appendix in DIEUDONNÉ (1972).

2 Proofs

2.1 Overview: Global structure of the proof

As we will see below, Theorem 1 follows immediately from its special case $n = 1$. For this case we need to fix an edge $e = \{v_0, v_1\}$, adjacent to the starting point v_0, and analyze the probability that this edge is traversed by the reinforced random walker in the first step. In principle, the choice of the edge e and the choice of the reference edge e_0 have nothing to do with each other. However, since the constants $z(a, v, e_0)$ do not depend on the choice of the reference edge e_0, one may choose e_0 such that $e_0 = \{v_0, v_1\} = e$. This makes the calculations slightly easier. Recall the notation $k(\pi)$ from Theorem 1. In the case $n = 1$, Theorem 1 can be rephrased as follows:

LEMMA 1 *For any edge $e_0 = \{v_0, v_1\}$ adjacent to the starting point v_0, one has*

$$\frac{a_{e_0}}{(Ia)_{v_0}} = \frac{z(a + k(v_0, v_1), v_1)}{z(a, v_0)}. \tag{13}$$

Using the fact

$$f_{a+k(v_0,v_1)}\, \mu_{v_1,e_0} = \frac{x_{e_0}}{y_{v_0}} f_a\, \mu_{v_0,e_0} \tag{14}$$

one can rewrite the claim (13) equivalently in the form

$$(Ia)_{v_0} \int_{\Omega_{e_0}} \frac{x_{e_0}}{y_{v_0}} f_a\, \mathrm{d}\mu_{v_0,e_0} = a_{e_0} \int_{\Omega_{e_0}} f_a\, \mathrm{d}\mu_{e_0,v_0}. \tag{15}$$

Equation (14) follows from the definition (5) of μ_{v_0,e_0} and the relation $f_{a+k(v_0,v_1)} = f_a \xi_{e_0}$. To prove Lemma 1, it suffices to show that

$$\int_{\Omega_{e_0}} \left((Ia)_{v_0} \frac{x_{e_0}}{y_{v_0}} - a_{e_0} \right) f_a\, \mathrm{d}\mu_{e_0,v_0} = 0. \tag{16}$$

This formula is proven by partial integration. More precisely, we use a partial integration with respect to a *Lie derivative* $\mathcal{L}_\varphi$ with a vector field φ, introduced in Section 2.3. This vector field is specified in Definition 2 below, such that

$$\mathcal{L}_\varphi(f_a) = \left((Ia)_{v_0} \frac{x_{e_0}}{y_{v_0}} - a_{e_0} \right) f_a \tag{17}$$

holds; see formula (38). The proof of formula (16) is then based on the following integration by parts:

$$\int_{\Omega_{e_0}} \mathcal{L}_\varphi(f_a)\, \mathrm{d}\mu_{e_0,v_0} = - \int_{\Omega_{e_0}} f_a\, \mathrm{d}\mathcal{L}_\varphi(\mu_{e_0,v_0}) \tag{18}$$

and the non-trivial fact $\mathcal{L}_\varphi(\mu_{e_0,v_0}) = 0$; see Lemma 9. Proving this formula requires some combinatoric preparations, which we describe in Section 2.4. It is also a non-trivial fact that the boundary terms in the partial integration vanish, as φ and μ_{e_0,v_0} both have singularities at the boundary. We deal with the boundary terms in Lemma 12 below.

2.2 The derivative of the map $(x_e)_e \mapsto (\log \xi_e)_e$

Recall the definition (6) of ξ_e. In the following, we use its logarithmic form

$$\log \xi_e = \log x_e - \frac{1}{2} \sum_{v \in V} I_{ve} \log y_v. \tag{19}$$

DEFINITION 1 *Let* $F=(F_{ef})_{e,f\in e_0^c}=X_{e_0^c}^{-1}-\frac{1}{2}R^\top Y^{-1}R$ *denote the Jacobi matrix of* $(\log\xi_e)_{e\in e_0^c}$ *with entries*

$$F_{ef}=\frac{\partial\log\xi_e}{\partial x_f}=\frac{\delta_{ef}}{x_e}-\frac{1}{2}\sum_{v\in V}\frac{I_{ve}I_{vf}}{y_v}. \tag{20}$$

Note that F is a symmetric matrix, indexed by *edges*. Closely related to the Jacobi matrix F is the matrix U, introduced in Section 1.3. Recall that it is labeled by *vertices* and fulfills the following identities:

$$U=U^\top=W+\Pi=2Y-RX_{e_0^c}R^\top=2Y-IXI^\top+\Pi. \tag{21}$$

The sum over trees in the "magic formula" can be expressed by various determinants. This is the content of the famous matrix-tree theorem:

LEMMA 2 (MATRIX-TREE THEOREM) *The determinant of* U *is given by*

$$\frac{1}{4}\det U=\det(W_{v_0^cv_0^c})=\sum_{T\in\mathcal{T}}\prod_{e\in T}x_e. \tag{22}$$

As a consequence, the matrices U *and* $W_{v_0^cv_0^c}$ *are invertible.*

Proof. Recall that σ denotes the signed incidence matrix of G. Since $\sum_{v\in V}\sigma_{ve}=0$ for all $e\in E$, and since $W=\sigma X\sigma^\top$, the sum over all rows in W and the sum over all columns in W vanish. Hence, $\det W=0$, and all entries of $\operatorname{ad}W$ are the same and equal to $\det(W_{v_0^cv_0^c})$. Since Π has rank 1 and precisely 4 non-vanishing entries 1, we conclude

$$\det U=\det(W+\Pi)=\det W+\sum_{u,v\in V}(\operatorname{ad}W)_{uv}\Pi_{vu}=4\det(W_{v_0^cv_0^c}). \tag{23}$$

The second equation in the claim (22) is the well-known matrix-tree theorem, see e.g. Theorem VI.29 in TUTTE (1984). ■

The following lemma describes two connections between the Jacobi matrix F and the matrix U. Both connections are fundamental for the proof of the "magic formula".

LEMMA 3 *a) The determinant of* F *is given by*

$$\det F=2^{-|V|}\Big(\prod_{e\in e_0^c}\frac{1}{x_e}\Big)\Big(\prod_{v\in V}\frac{1}{y_v}\Big)\det U. \tag{24}$$

As a consequence, the matrix F *is invertible.*

b) For any column vector $s\in\mathbb{R}^{e_0^c}$*, one has*

$$R_{v_0}F^{-1}s=2y_{v_0}(U^{-1}RX_{e_0^c}s)_{v_0}. \tag{25}$$

Proof. We prove a) and b) simultaneously. Given a fixed column vector $s \in \mathbb{R}^{e_0^c}$, consider the affine linear map $h : \mathbb{R} \to \mathbb{R}$, $h(t) = \det F + tR_{v_0} \operatorname{ad}(F)s$. We calculate

$$\begin{aligned} h(t) =& \det(F + tsR_{v_0}) = \det(X_{e_0^c}^{-1} - \frac{1}{2}R^\top Y^{-1}R + tsR_{v_0}) \\ =& \det\left[X_{e_0^c}^{-1} - \frac{1}{2}(R^\top, s)\begin{pmatrix} Y^{-1} & 0 \\ 0 & -2t \end{pmatrix}\begin{pmatrix} R \\ R_{v_0} \end{pmatrix}\right]. \end{aligned} \tag{26}$$

Now, for any invertible $n \times n$ matrix A, any $n \times m$ matrix B, any invertible $m \times m$ matrix C, and any $m \times n$ matrix D, we have

$$\det(A + BCD) = \det A \det C \det(DA^{-1}B + C^{-1}). \tag{27}$$

We sketch a proof of (27): Writing the left hand side as $\det A \det(\mathrm{Id}_n + (A^{-1}B)(CD))$ and the right hand side as $\det A \det(\mathrm{Id}_m + (CD)(A^{-1}B))$, we reduce the claim (27) to the special case $A = \mathrm{Id}_n$, $C = \mathrm{Id}_m$, i.e. $\det(\mathrm{Id}_n + BD) = \det(\mathrm{Id}_m + DB)$. Now, expanding the determinants and using the Cauchy-Binet formula in the second equality (see e.g. Proposition 2.1.2 in SERRE (2002)), the claim follows: $\det(\mathrm{Id}_n + BD) = \sum_{J \subseteq \{1,\dots,n\}} \det((BD)_{JJ})$ $= \sum_{K \subseteq \{1,\dots,m\}} \det((DB)_{KK}) = \det(\mathrm{Id}_m + DB)$.
Using formula (27), we conclude for $t \neq 0$:

$$h(t) = \det(X_{e_0^c}^{-1}) \det(Y^{-1})(-2t) \det\left[-\frac{1}{2}\begin{pmatrix} R \\ R_{v_0} \end{pmatrix} X_{e_0^c}(R^\top, s) + \begin{pmatrix} Y & 0 \\ 0 & -\frac{1}{2t} \end{pmatrix}\right]. \tag{28}$$

We denote the v_0-th row of Y by Y_{v_0}. Subtracting the v_0-th row of the argument of the last determinant from the last row, the last determinant equals

$$\begin{aligned} &\det\left[-\frac{1}{2}\begin{pmatrix} R \\ 0 \end{pmatrix} X_{e_0^c}(R^\top, s) + \begin{pmatrix} Y & 0 \\ -Y_{v_0} & -\frac{1}{2t} \end{pmatrix}\right] \\ =&2^{-|V|-1} \det\begin{pmatrix} 2Y - RX_{e_0^c}R^\top & -RX_{e_0^c}s \\ -2Y_{v_0} & -\frac{1}{t} \end{pmatrix} = 2^{-|V|-1} \det\begin{pmatrix} U & -RX_{e_0^c}s \\ -2Y_{v_0} & -\frac{1}{t} \end{pmatrix} \\ =&2^{-|V|-1}\left[-\frac{1}{t}\det U - 2Y_{v_0} \operatorname{ad}(U)RX_{e_0^c}s\right]. \end{aligned} \tag{29}$$

In the last step, we used that for any $n \times n$ matrix A, any column vector b of length n, any row vector c of length n, and any scalar d, one has

$$\det\begin{pmatrix} A & b \\ c & d \end{pmatrix} = d \det A - c \operatorname{ad}(A)b. \tag{30}$$

This can be seen by expanding the determinant on the left hand side with respect to the last row and then with respect to the last column.

Substituting (29) in (28) and using the definition of $h(t)$, this yields for all $t \neq 0$

$$\det F + tR_{v_0} \operatorname{ad}(F)s = 2^{-|V|} \det(X_{e_0^c}^{-1}) \det(Y^{-1}) \left[\det U + 2tY_{v_0} \operatorname{ad}(U)RX_{e_0^c}s\right]. \tag{31}$$

Since this identity is valid for all $t \neq 0$, we conclude

$$\det F = 2^{-|V|} \det(X_{e_0^c}^{-1}) \det(Y^{-1}) \det U \quad \text{and} \tag{32}$$

$$R_{v_0} \operatorname{ad}(F) s = 2^{-|V|} \det(X_{e_0^c}^{-1}) \det(Y^{-1}) 2 Y_{v_0} \operatorname{ad}(U) R X_{e_0^c} s. \tag{33}$$

Claim a) of the lemma follows from (32). Since $\det U \neq 0$ by Lemma 2, it follows that $\det F \neq 0$. In particular, F is invertible. Hence, we can divide equation (33) by (32). This proves part b) of the lemma. ∎

2.3 The vector field φ and the corresponding Lie derivative

The following vector field plays a key role in the proof:

DEFINITION 2 *We define the vector field* $\varphi = (\varphi_e)_{e \in e_0^c} : \Omega_{e_0} \to \mathbb{R}^{e_0^c}$ *by*

$$\varphi = \frac{1}{y_{v_0}} F^{-1} R_{v_0}^\top. \tag{34}$$

For our arguments, we need the Lie derivative $\mathcal{L}_\varphi \omega$ with respect to the vector field φ for arguments ω of various types: real-valued functions, 1-forms, and measures having a smooth density, all living on Ω_{e_0}. It is convenient to identify these measures with $(\dim \Omega_{e_0})$-forms. When we do this, we drop the label "d" in integrals, in order to avoid confusion with the exterior derivative "*d*". The vector field φ induces a *flow* on Ω_{e_0}. Roughly speaking, the Lie derivative $\mathcal{L}_\varphi \omega$ is defined as the time derivative of ω *transported with the flow* at time 0, whatever the type of ω is. For example, if g is a smooth real-valued function, then its Lie derivative equals the directional derivative

$$\mathcal{L}_\varphi g = \varphi^\top \nabla g \tag{35}$$

in direction of φ, and if ω is the Lebesgue measure (volume form), then its Lie derivative equals $\mathcal{L}_\varphi \omega = (\operatorname{div} \varphi) \omega$. The Lie derivative obeys similar properties as the usual derivative, such as the chain rule and product rules like $\mathcal{L}_\varphi(\omega_1 \wedge \ldots \wedge \omega_n) = \sum_{i=1}^n \omega_1 \wedge \ldots \wedge \omega_{i-1} \wedge (\mathcal{L}_\varphi \omega_i) \wedge \omega_{i+1} \wedge \ldots \wedge \omega_n$ for 1-forms $\omega_1, \ldots, \omega_n$ and $\mathcal{L}_\varphi(g\mu) = \mathcal{L}_\varphi(g)\mu + g\mathcal{L}_\varphi(\mu)$ for smooth real-valued functions g and signed measures ($(\dim \Omega_{e_0})$-forms) μ having a smooth density with respect to the Lebesgue measure. For a proof of these (and more general) product rules, see e.g. DIEUDONNÉ (1972), formula (17.14.7.5). Whenever the function g and the density of μ are nonzero, we can write the last product rule also in logarithmic form

$$\frac{\mathcal{L}_\varphi(g\mu)}{g\mu} = \frac{\mathcal{L}_\varphi(g)}{g} + \frac{\mathcal{L}_\varphi(\mu)}{\mu}, \tag{36}$$

where the quotient of measures means the Radon-Nikodym derivative. For more background information on the Lie derivative and the exterior calculus, see e.g. DIEUDONNÉ (1972), Sections XVII 14 and XVII 15.

The following lemma collects the Lie derivatives of some important terms. Recall the definition (7) of f_a.

LEMMA 4 *One has*

$$\mathcal{L}_\varphi(\log\xi_e) = \frac{I_{v_0 e}}{y_{v_0}} - \delta_{ee_0} \qquad \textit{for all } e\in E \textit{ and} \tag{37}$$

$$\frac{\mathcal{L}_\varphi f_a}{f_a} = (Ia)_{v_0}\frac{x_{e_0}}{y_{v_0}} - a_{e_0}. \tag{38}$$

Proof. We consider first the case $e \in e_0^c$. In this case, using formula (35) and Definitions 1 and 2, we get

$$\mathcal{L}_\varphi(\log\xi_e) = \sum_{f\in e_0^c}\varphi_f\frac{\partial\log\xi_e}{\partial x_f} = \sum_{f\in e_0^c}\varphi_f F_{ef} = (F\varphi)_e = \frac{(FF^{-1}R_{v_0}^\top)_e}{y_{v_0}} = \frac{R_{v_0 e}}{y_{v_0}} = \frac{I_{v_0 e}}{y_{v_0}}. \tag{39}$$

This proves the claim (37) in the case $e \in e_0^c$. Next, we consider the case $e = e_0$. Let $f \in e_0^c$. We calculate, using the expression (19) for $\log\xi_g$, $g\in E$:

$$\begin{aligned}\sum_{g\in E}x_g\frac{\partial\log\xi_g}{\partial x_f} &= \sum_{g\in E}x_g\frac{\partial}{\partial x_f}\left(\log x_g - \frac12\sum_{v\in V}I_{vg}\log y_v\right) = \sum_{g\in E}x_g\left(\frac{\delta_{gf}}{x_g} - \frac12\sum_{v\in V}\frac{I_{vg}I_{vf}}{y_v}\right)\\ &= 1 - \frac12\sum_{v\in V}\frac{I_{vf}}{y_v}\sum_{g\in E}I_{vg}x_g = 1 - \frac12\sum_{v\in V}I_{vf} = 1-\frac12\cdot 2 = 0.\end{aligned} \tag{40}$$

Using the normalization $x_{e_0}=1$, this yields

$$\frac{\partial\log\xi_{e_0}}{\partial x_f} = -\sum_{g\in e_0^c}x_g\frac{\partial\log\xi_g}{\partial x_f} \tag{41}$$

and hence, using (35) in the first and second step and (39) in the third step:

$$\begin{aligned}\mathcal{L}_\varphi(\log\xi_{e_0}) &= \sum_{f\in e_0^c}\varphi_f\frac{\partial\log\xi_{e_0}}{\partial x_f} = -\sum_{g\in e_0^c}x_g\mathcal{L}_\varphi(\log\xi_g)\\ &= -\sum_{g\in e_0^c}\frac{I_{v_0 g}x_g}{y_{v_0}} = \frac{I_{v_0 e_0}}{y_{v_0}} - \sum_{g\in E}\frac{I_{v_0 g}x_g}{y_{v_0}} = \frac{I_{v_0 e_0}}{y_{v_0}} - 1.\end{aligned} \tag{42}$$

Thus, the claim (37) is also proven in the case $e = e_0$. To prove the second claim (38), we take the logarithm of the defining equation (7) of f_a: $\log f_a = \sum_{e\in E}a_e\log\xi_e$. Taking the Lie derivative of this equation and using the chain rule and the first claim (37), we obtain the second claim (38) as follows:

$$\frac{\mathcal{L}_\varphi f_a}{f_a} = \mathcal{L}_\varphi(\log f_a) = \sum_{e\in E}a_e\mathcal{L}_\varphi(\log\xi_e) = \sum_{e\in E}a_e\left(\frac{I_{v_0 e}}{y_{v_0}} - \delta_{ee_0}\right) = (Ia)_{v_0}\frac{x_{e_0}}{y_{v_0}} - a_{e_0}. \tag{43}$$

■

The following formula describes the Lie derivative of scalar-valued functions in a convenient way. It relies on the connection between the matrices F and U.

LEMMA 5 (EXPLICIT FORMULA FOR THE LIE DERIVATIVE) *For any smooth function* $g : \Omega_{e_0} \to \mathbb{R}$*, one has* $\mathcal{L}_\varphi g = 2(U^{-1}RX_{e_0^c}\nabla g)_{v_0}$.

Proof. Using formula (35), the definition (34) of φ, the symmetry $F^\top = F$ of F, and formula (25) from Lemma 3b), the claim follows immediately:

$$\mathcal{L}_\varphi g = \varphi^\top \nabla g = \frac{1}{y_{v_0}} R_{v_0} F^{-1} \nabla g = 2(U^{-1}RX_{e_0^c}\nabla g)_{v_0}. \qquad \blacksquare \tag{44}$$

2.4 The Lie derivative of the tree term and other determinants

The following combinatoric lemma relies on the fact that whenever three vertices are adjacent to the same edge, then at least two of them are equal.

LEMMA 6 (TRIPLE INCIDENCE FORMULA) *For all* $u, v, w \in V$*, one has*

$$\sum_{e \in e_0^c} R_{ue}R_{ve}R_{we}x_e = 1_{\{u=v=w\}}(4y_v + 2I_{ve_0}) - \delta_{uv}U_{vw} - \delta_{vw}U_{wu} - \delta_{wu}U_{uv}. \tag{45}$$

Proof. Note that the formula (45) is symmetric under permutations of u, v, w. We distinguish 3 cases:

- If the three vertices u, v, and w are pairwise different, then the left hand side in (45) vanishes, since no edge can be incident to three different vertices. The right hand side in (45) vanishes also in this case.

- Assume that two of the three vertices u, v, and w are equal, but different from the third one. Using symmetry, without loss of generality, we assume $u = v \neq w$. Because for any edge e, we have $R_{ue} \in \{0, 1\}$, we conclude $R_{ue}R_{ve} = R_{ue}^2 = R_{ue}$. Thus, the left hand side in (45) equals
$$\sum_{e \in e_0^c} R_{ue}R_{we}x_e = (RX_{e_0^c}R^\top)_{uw} = -U_{uw} \tag{46}$$
where we use formula (21) and the consequence $Y_{uw} = 0$ of $u \neq w$ in the last step. Since $-U_{uw}$ equals the right hand side in (45), formula (45) follows in the case $u = v \neq w$.

- In the case $u = v = w$, we get $R_{ue}R_{ve}R_{we} = R_{ve} = I_{ve}$ for any edge e. Thus, using $x_{e_0} = 1$ and $U_{vv} = y_v + I_{ve_0}$, the left hand side in (45) equals
$$\sum_{e \in e_0^c} R_{ve}x_e = \sum_{e \in E} I_{ve}x_e - I_{ve_0} = y_v - I_{ve_0} = 4y_v + 2I_{ve_0} - 3U_{vv}, \tag{47}$$
which equals also the right hand side in (45). $\blacksquare$

This combinatoric lemma is the key to the next lemma, which deals with the Lie derivative of the sum over spanning trees, written by the matrix-tree theorem Lemma 2 as a determinant:

LEMMA 7 *The following identity holds:* $\mathcal{L}_\varphi \log\det(W_{v_0^c v_0^c}) = \sum_{v\in v_0^c} \mathcal{L}_\varphi \log y_v$.

Proof. We determine first the Lie derivatives of some ingredient terms. From Lemma 5, we know

$$\mathcal{L}_\varphi x_e = 2\sum_{u\in V} U^{-1}_{v_0u} R_{ue} x_e, \quad (e\in e_0^c). \tag{48}$$

Using $\mathcal{L}_\varphi x_{e_0} = 0$ and (21), this implies for $v \in V$:

$$\begin{aligned}\mathcal{L}_\varphi y_v &= \sum_{e\in e_0^c} R_{ve}\mathcal{L}_\varphi x_e = 2\sum_{u\in V} U^{-1}_{v_0u} \sum_{e\in e_0^c} R_{ue} x_e R_{ve} = 2\sum_{u\in V} U^{-1}_{v_0u}(2\delta_{uv} y_v - U_{uv}) \\ &= 4U^{-1}_{v_0v} y_v - 2\delta_{vv_0}.\end{aligned} \tag{49}$$

In the case $v \in v_0^c$, the term δ_{vv_0} vanishes. Using the chain rule, we conclude in this case:

$$\mathcal{L}_\varphi \log y_v = \frac{\mathcal{L}_\varphi y_v}{y_v} = 4U^{-1}_{v_0v}, \quad (v\in v_0^c). \tag{50}$$

Next, we determine the Lie derivative of the left side in the claim of Lemma 7, using the chain rule again. Note that whenever A_z is a smooth family of invertible matrices, then

$$\frac{d}{dz}\log\det A_z = \operatorname{trace}\Big(A_z^{-1}\frac{d}{dz}A_z\Big). \tag{51}$$

In the following formula, the Lie derivative of a matrix should be read componentwise:

$$\mathcal{L}_\varphi \log\det(W_{v_0^c v_0^c}) = \operatorname{trace}[(W_{v_0^c v_0^c})^{-1}\mathcal{L}_\varphi W_{v_0^c v_0^c}] = \operatorname{trace}[(W_{v_0^c v_0^c})^{-1}\mathcal{L}_\varphi U_{v_0^c v_0^c}]. \tag{52}$$

In the last step, we used that $U = W + \Pi$ by (21), and that the Lie derivative of the constant matrix Π vanishes. Recall that $e_0 = \{v_0, v_1\}$. We calculate the Lie derivatives of the relevant entries of U: Let $u, v \in v_0^c$. Using (48), (49), the triple-incidence formula (45), and the facts $\delta_{vv_0} = 0$ and $I_{ve_0} = \delta_{vv_0} + \delta_{vv_1} = \delta_{vv_1}$ for $v \in v_0^c$, we get:

$$\begin{aligned}\mathcal{L}_\varphi U_{uv} &= \mathcal{L}_\varphi[2\delta_{uv}y_v - \sum_{e\in e_0^c} R_{ue}x_e R_{ve}] = 8\delta_{uv}U^{-1}_{v_0v}y_v - 2\sum_{w\in V} U^{-1}_{v_0w}\sum_{e\in e_0^c} R_{ue}R_{ve}R_{we}x_e \\ &= 8\delta_{uv}U^{-1}_{v_0v}y_v - 2\sum_{w\in V} U^{-1}_{v_0w}[1_{\{u=v=w\}}(4y_v + 2I_{ve_0}) - \delta_{uv}U_{vw} - \delta_{vw}U_{wu} - \delta_{wu}U_{uv}] \\ &= 8\delta_{uv}U^{-1}_{v_0v}y_v - 2\delta_{uv}U^{-1}_{v_0v}(4y_v + 2I_{ve_0}) + 2\delta_{uv}\delta_{vv_0} + 2U^{-1}_{v_0v}U_{vu} + 2U^{-1}_{v_0u}U_{uv} \\ &= -4\delta_{uv}U^{-1}_{v_0v_1}\delta_{vv_1} + 2(U^{-1}_{v_0v} + U^{-1}_{v_0u})U_{uv} = -4\delta_{uv}U^{-1}_{v_0v_1}\delta_{vv_1} + 2(U^{-1}_{v_0v} + U^{-1}_{v_0u})(W_{uv} + \Pi_{uv}) \\ &= -4\delta_{uv}U^{-1}_{v_0v_1}\delta_{vv_1} + 2(U^{-1}_{v_0v} + U^{-1}_{v_0u})W_{uv} + 4U^{-1}_{v_0v_1}\delta_{uv_1}\delta_{vv_1}.\end{aligned} \tag{53}$$

In the last step, we used that $\Pi_{v_0^c v_0^c}$ has a single nonzero entry 1, namely at the index pair $(u, v) = (v_1, v_1)$. When we substitute this in (52), we obtain, using the symmetry $W = W^\top$:

$$\begin{aligned}\mathcal{L}_\varphi \log\det(W_{v_0^c v_0^c}) &= \sum_{u,v\in v_0^c} (W^{-1}_{v_0^c v_0^c})_{vu}(-4\delta_{uv}U^{-1}_{v_0v_1}\delta_{vv_1} + 2(U^{-1}_{v_0v} + U^{-1}_{v_0u})W_{uv} + 4U^{-1}_{v_0v_1}\delta_{uv_1}\delta_{vv_1}) \\ &= -4(W^{-1}_{v_0^c v_0^c})_{v_1v_1}U^{-1}_{v_0v_1} + 4\sum_{u,v\in v_0^c}(W^{-1}_{v_0^c v_0^c})_{vu}U^{-1}_{v_0v}W_{uv} + 4(W^{-1}_{v_0^c v_0^c})_{v_1v_1}U^{-1}_{v_0v_1} = 4\sum_{v\in v_0^c} U^{-1}_{v_0v}.\end{aligned} \tag{54}$$

When we compare this equation with the sum of (50) over $v \in v_0^c$, Lemma 7 follows. ■

The logarithmic Lie derivative of the volume form in the $(\log \xi_e)_e$ coordinates is determined in the following lemma:

LEMMA 8 *The following identity holds:* $\mathcal{L}_\varphi\Big(\bigwedge_{e\in e_0^c} d\log\xi_e \Big) = -\mathcal{L}_\varphi(\log y_{v_0}) \bigwedge_{e\in e_0^c} d\log\xi_e.$

Proof. Using the product rule, we calculate:

$$\mathcal{L}_\varphi\Big(\bigwedge_{e\in e_0^c} d\log\xi_e \Big) = \sum_{f\in e_0^c} \mathrm{sign}_f\, \mathcal{L}_\varphi(d\log\xi_f) \wedge \bigwedge_{e\in(e_0 f)^c} d\log\xi_e, \tag{55}$$

where sign_f means the sign required to move the first factor to the f-th position in the wedge product. The Lie derivative and the exterior derivative d commute; see DIEUDONNÉ (1972), formula (17.14.10.1) and, more generally, formula (17.15.3.3). We conclude for $f \in e_0^c$, using equation (37) twice:

$$\mathcal{L}_\varphi(d\log\xi_f) = d(\mathcal{L}_\varphi(\log\xi_f)) = d\frac{I_{v_0 f}}{y_{v_0}} = -\frac{I_{v_0 f}}{y_{v_0}}\, d\log y_{v_0} = -\mathcal{L}_\varphi(\log\xi_f)\, d\log y_{v_0}. \tag{56}$$

Since the Jacobi matrix $F = (\partial\log\xi_e/\partial x_f)_{e,f\in e_0^c}$ is invertible, we can write $d\log y_{v_0}$ uniquely as a linear combination

$$d\log y_{v_0} = \sum_{g\in e_0^c} \alpha_g\, d\log\xi_g. \tag{57}$$

Substituting this in (56) and then in (55), we obtain, using (57) once more:

$$\begin{aligned}\mathcal{L}_\varphi\Big(\bigwedge_{e\in e_0^c} d\log\xi_e \Big) &= -\sum_{f\in e_0^c} \mathrm{sign}_f\, \mathcal{L}_\varphi(\log\xi_f) \sum_{g\in e_0^c} \alpha_g\, d\log\xi_g \wedge \bigwedge_{e\in(e_0 f)^c} d\log\xi_e \\ &= -\sum_{f\in e_0^c} \mathcal{L}_\varphi(\log\xi_f)\alpha_f \bigwedge_{e\in e_0^c} d\log\xi_e = -\mathcal{L}_\varphi(\log y_{v_0}) \bigwedge_{e\in e_0^c} d\log\xi_e. \quad ■ \end{aligned} \tag{58}$$

2.5 Proof of Theorem 1

All calculations of the previous sections culminate in the following lemma:

LEMMA 9 *One has* $\mathcal{L}_\varphi \mu_{v_0,e_0} = 0$.

Proof. Using Lemma 2, the normalization $x_{e_0} = 1$, and the relation (24) between $\det U$ and $\det F$, we rewrite the defining equations (5) of μ_{v_0,e_0} as

$$\begin{aligned}\mu_{v_0,e_0} &= \frac{1}{2}\sqrt{y_{v_0}}\left(\prod_{v\in V} y_v{}^{-1/2}\right)\sqrt{\det U}\left(\prod_{e\in E}\frac{1}{x_e}\right)\bigwedge_{e\in e_0^c} dx_e \\ &= \frac{1}{2}\sqrt{\frac{\prod_{v\in v_0^c} y_v}{\det U}}y_{v_0}\left(\prod_{e\in E}\frac{1}{x_e}\right)\left(\prod_{v\in V}\frac{1}{y_v}\right)\det U\bigwedge_{e\in e_0^c} dx_e \\ &= 2^{|V|-1}\sqrt{\frac{\prod_{v\in v_0^c} y_v}{\det U}}y_{v_0}\det F\bigwedge_{e\in e_0^c} dx_e = 2^{|V|-2}\sqrt{\frac{\prod_{v\in v_0^c} y_v}{\det(W_{v_0^c v_0^c})}}y_{v_0}\det F\bigwedge_{e\in e_0^c} dx_e. \end{aligned} \tag{59}$$

Now, $\det F$ is the Jacobi determinant of the map $(x_e)_{e\in e_0^c} \mapsto (\log \xi_e)_{e\in e_0^c}$. Using the definition of the wedge product with determinants from Section 1.3, this yields

$$\det F \bigwedge_{e\in e_0^c} dx_e = \bigwedge_{e\in e_0^c} d\log\xi_e. \tag{60}$$

When we substitute this in (59), we get

$$\mu_{v_0,e_0} = 2^{|V|-2}\sqrt{\frac{\prod_{v\in v_0^c} y_v}{\det(W_{v_0^c v_0^c})}}y_{v_0}\bigwedge_{e\in e_0^c} d\log\xi_e. \tag{61}$$

Using (36), we take the logarithmic Lie derivative of this equation:

$$\frac{\mathcal{L}_\varphi \mu_{v_0,e_0}}{\mu_{v_0,e_0}} = \frac{1}{2}\mathcal{L}_\varphi\left(\sum_{v\in v_0^c}\log y_v - \log\det(W_{v_0^c v_0^c})\right) + \mathcal{L}_\varphi(\log y_{v_0}) + \frac{\mathcal{L}_\varphi\Big(\bigwedge_{e\in e_0^c} d\log\xi_e\Big)}{\bigwedge_{e\in e_0^c} d\log\xi_e}. \tag{62}$$

By Lemma 7, the first Lie derivative on the right hand side vanishes. By Lemma 8, the two remaining terms cancel each other. Thus everything on the right hand side in the last equation cancels. This proves the lemma. ■

Using this lemma, we see that the whole integrand in formula (16) can be written as a Lie derivative:

LEMMA 10 *For all $a \in (0,\infty)^E$, one has* $\left((Ia)_{v_0}\frac{x_{e_0}}{y_{v_0}} - a_{e_0}\right) f_a\mu_{v_0,e_0} = \mathcal{L}_\varphi(f_a\mu_{v_0,e_0})$.

Proof. Using formula (38) and $\mathcal{L}_\varphi(\mu_{v_0,e_0}) = 0$ from Lemma 9, the following calculation implies the claim of the lemma:

$$\begin{aligned}&\left((Ia)_{v_0}\frac{x_{e_0}}{y_{v_0}} - a_{e_0}\right) f_a\mu_{v_0,e_0} = \mathcal{L}_\varphi(f_a)\mu_{v_0,e_0} \\ &= \mathcal{L}_\varphi(f_a)\mu_{v_0,e_0} + f_a\mathcal{L}_\varphi(\mu_{v_0,e_0}) = \mathcal{L}_\varphi(f_a\mu_{v_0,e_0}). \qquad ■\end{aligned} \tag{63}$$

The following technical lemma will be the key to treat the boundary terms on the one hand, and to prove integrability of $f_a\mu_{v_0,e_0}$ on the other hand:

LEMMA 11 *We abbreviate $\Sigma = \sum_{e\in E} x_e$. There are strictly positive constants c_1, c_2, and c_3, depending only on the graph G, such that the following bound holds on Ω_{e_0}:*

$$\prod_{e\in E}\xi_e \le c_1\prod_{e\in E}\left(\frac{x_e}{\Sigma}\right)^{c_2} \le c_1\prod_{e\in e_0^c}\frac{x_e^{c_2}}{(1+x_e)^{c_2+c_3}}. \tag{64}$$

Furthermore, one has

$$\Big(\prod_{v\in v_0^c}\frac{1}{y_v}\Big)\sum_{T\in\mathcal{T}}\prod_{e\in T}x_e \le |\mathcal{T}|. \tag{65}$$

Proof. Let us abbreviate $\log x_e = \ell_e$, $\ell = (\ell_e)_{e\in E}$. Using that $y_v \ge x_e$ for all $v\in V$ and $e\in E$ with $I_{ve}=1$, the logarithm of the left hand side in (64) can be bounded as follows:

$$\begin{aligned}&\sum_{e\in E}\log\xi_e = \sum_{e\in E}\left(\ell_e - \frac{1}{2}\sum_{v\in V} I_{ve}\log y_v\right)\\ &\le \sum_{e\in E}\left(\ell_e - \frac{1}{2}\sum_{v\in V}\max_{f\in E:\, I_{vf}=1}\ell_f\right) = \frac{1}{2}\sum_{v\in V}\sum_{e\in E} I_{ve}\left(\ell_e - \max_{f\in E:\, I_{vf}=1}\ell_f\right) \stackrel{\text{def}}{=} h_1(\ell)\le 0.\end{aligned} \tag{66}$$

The last sum $h_1(\ell)$ vanishes only if for all vertices v, all edges e adjacent to v have the same logarithmic weight ℓ_e. Since G is connected and since we use the normalization $\ell_{e_0}=0$, this can only happen if $\ell = 0$.

On the other hand, using $\Sigma \le |E|\max_{f\in E} x_f$, we conclude

$$\log\prod_{e\in E}\frac{x_e}{\Sigma} \ge -|E|\log|E| + \sum_{e\in E}(\ell_e - \max_{f\in E}\ell_f) \stackrel{\text{def}}{=} -|E|\log|E| + h_2(\ell). \tag{67}$$

We have also $h_2(\ell)\le 0$. Using the normalization $\ell_{e_0}=0$ again, $h_2(\ell)=0$ occurs also only if $\ell = 0$. Furthermore, we have for any $\ell\ne 0$ with $\ell_{e_0}=0$:

$$\frac{h_1(\ell)}{h_2(\ell)} = \frac{h_1(\ell/\|\ell\|_2)}{h_2(\ell/\|\ell\|_2)} > 0. \tag{68}$$

Since h_1 and h_2 are continuous functions, using a compactness argument on the unit sphere, the last expression is bounded from below by a positive constant c_2. We combine this with (66) and (67). We set $c_1 = |E|^{c_2|E|}$. For the second inequality in the following estimate, we use that $h_2\le 0$:

$$\log\prod_{e\in E}\xi_e \le h_1(\ell) \le c_2 h_2(\ell) \le \log c_1 + c_2\log\prod_{e\in E}\frac{x_e}{\Sigma}. \tag{69}$$

This yields the first inequality in the claim (64). Setting $c_3 = 1/(|E|-1)$, the second claim in (64) follows immediately from the facts $x_{e_0}=1$ and $\Sigma \ge 1 + x_e$ for any $e\in e_0^c$.

To prove (65), let $T\in\mathcal{T}$ be a spanning tree. For every $e=\{u_1,u_2\}\in T$, let $v(e)\in\{u_1,u_2\}$ denote the endpoint of e with the larger distance in T from v_0. Then,

$$\Big(\prod_{v\in v_0^c}\frac{1}{y_v}\Big)\prod_{e\in T}x_e = \prod_{e\in T}\frac{x_e}{y_{v(e)}} \le 1. \tag{70}$$

Summing this over all $T \in \mathcal{T}$ proves the claim (65). ■

The last lemma in this paper proves integrability of $f_a \mu_{v_0,e_0}$. It also makes the partial integration explicit and deals with the boundary terms, and relies on Stokes' theorem. We derive the version of Stokes' theorem that we use directly from the fundamental theorem of calculus. For more general background information on Stokes' theorem, see e.g. DIEUDONNÉ (1972), Section XVI 24.

LEMMA 12 *For all initial weights $a \in (0,\infty)^E$, we have $z(a, v_0) < \infty$ and*

$$\int_{\Omega_{e_0}} \left((Ia)_{v_0} \frac{x_{e_0}}{y_{v_0}} - a_{e_0} \right) f_a \mu_{v_0,e_0} = \int_{\Omega_{e_0}} \mathcal{L}_\varphi (f_a \mu_{v_0,e_0}) = 0. \tag{71}$$

Proof of Lemma 12 and Lemma 1. For given $a \in (0,\infty)^E$, we set $m = m(a) = \min_{e \in E} a_e > 0$. Since $\xi_e \le 1$ for all $e \in E$, the inequality (64) implies

$$f_a \le \prod_{e \in E} \xi_e^m \le {c_1}^m \prod_{e \in e_0^c} \frac{x_e^{c_2 m}}{(1 + x_e)^{(c_2 + c_3) m}}. \tag{72}$$

Combining this with the definition (8) of $z(a, v_0)$, the definition (5) of μ_{v_0,e_0}, and the bound (65) from Lemma 11 yields

$$\begin{aligned} z(a, v_0) = \int_{\Omega_{e_0}} f_a \, \mathrm{d}\mu_{v_0,e_0} &= \int_{\Omega_{e_0}} f_a \cdot \sqrt{\Big(\prod_{v \in V} \frac{1}{y_v}\Big) \sum_{T \in \mathcal{T}} \prod_{e \in T} x_e} \prod_{e \in e_0^c} \frac{\mathrm{d}x_e}{x_e} \\ &\le {c_1}^m \sqrt{|\mathcal{T}|} \prod_{e \in e_0^c} \int_0^\infty \frac{x_e^{c_2 m - 1}}{(1 + x_e)^{(c_2 + c_3) m}} \, \mathrm{d}x_e < \infty. \end{aligned} \tag{73}$$

Note that the last integral is finite since its integrand is integrable close to 0 and close to ∞.

By Lemma 10, the two integrals in (71) are equal, if any of them is finite. Let $S(a)$ denote the left-hand side of (71). Since $(Ia)_{v_0} x_{e_0} / y_{v_0} - a_{e_0}$ is a bounded function on Ω_{e_0}, the integral $S(a)$ is finite. Even more, using that $0 < \xi_e \le 1$, one sees that $S(a)$ is a real-analytic function of $a \in (0,\infty)^E$. As a consequence, by the identity theorem for real analytic functions, S vanishes on Ω_{e_0} if it vanishes on an open non-empty subset of Ω_{e_0}. We show now that $S(a) = 0$ whenever $m(a)$ is large enough. Consider a large box $\Omega_N = [1/N, N]^{e_0^c} \subset \Omega_{e_0}$. We get

$$\int_{\Omega_{e_0}} \mathcal{L}_\varphi (f_a \mu_{v_0,e_0}) = \lim_{N \to \infty} \int_{\Omega_N} \mathcal{L}_\varphi (f_a \mu_{v_0,e_0}). \tag{74}$$

Let ρ_{v_0,e_0} denote the density of μ_{v_0,e_0} with respect to the Lebesgue measure on Ω_{e_0}. Next we apply Stokes' theorem. In concrete terms, deriving the version of Stokes' theorem we need, we calculate:

$$\int_{\Omega_N} \mathcal{L}_\varphi(f_a\mu_{v_0,e_0}) = \int_{\Omega_N} [\mathcal{L}_\varphi(f_a\rho_{v_0,e_0}) + \operatorname{div}(\varphi) f_a\rho_{v_0,e_0}] \prod_{f\in e_0^c} \mathrm{d}x_f$$
$$= \sum_{e\in e_0^c} \int_{\Omega_N} \left[\varphi_e \frac{\partial}{\partial x_e}(f_a\rho_{v_0,e_0}) + \frac{\partial\varphi_e}{\partial x_e} f_a\rho_{v_0,e_0}\right] \prod_{f\in e_0^c} \mathrm{d}x_f = \sum_{e\in e_0^c} \int_{\Omega_N} \frac{\partial}{\partial x_e}[\varphi_e f_a\rho_{v_0,e_0}] \prod_{f\in e_0^c} \mathrm{d}x_f$$
$$= \sum_{e\in e_0^c} \int_{[1/N,N]^{(ee_0)^c}} [\varphi_e f_a\rho_{v_0,e_0}]_{x_e=1/N}^{x_e=N} \prod_{f\in(ee_0)^c} \mathrm{d}x_f. \tag{75}$$

Next we estimate $\varphi_e f_a\rho_{v_0,e_0}$ roughly on the boundary $\partial\Omega_N$ of the box Ω_N. We define $r : \Omega_{e_0} \to \mathbb{R}$,

$$r(x) = \prod_{e\in e_0^c} [(1+x_e)(1+x_e^{-1})]. \tag{76}$$

Let Ξ denote the vector space of all functions $h : \Omega_{e_0}^c \to \mathbb{R}$ with the property that there exist $c_4(h) > 0$ and $c_5(h) > 0$ such that

$$|h| \le c_4(h) r^{c_5(h)}. \tag{77}$$

We show now that $\varphi_e\rho_{v_0,e_0} \in \Xi$. Note that Ξ contains all x_e and $1/x_e$, $e \in E$, and is closed under addition and multiplication. By the definition (34) of φ and formula (25), we have $\varphi_e = 2(\det U)^{-1}(\operatorname{ad}(U)RX_{e_0}^c)_{v_0e}$. Now for any spanning tree $T \in \mathcal{T}$, using the matrix-tree theorem (Lemma 2), we have $\det U \ge 4\prod_{e\in T} x_e$, which implies $(\det U)^{-1} \in \Xi$. It follows that $\varphi_e \in \Xi$ for any $e \in e_0^c$. Bounding the tree term in definition (5) by (65), one sees that $\rho_{v_0,e_0} \in \Xi$ and thus $\varphi_e\rho_{v_0,e_0} \in \Xi$.

We claim that for any given $h \in \Xi$, $h \ge 0$, there exists $m_0(h)$ such that for all $a \in (0,\infty)^E$ with $m(a) \ge m_0(h)$, one has

$$\sum_{e\in e_0^c} \int_{[1/N,N]^{(ee_0)^c}} [f_a h]_{x_e=1/N}^{x_e=N} \prod_{f\in(ee_0)^c} \mathrm{d}x_f \stackrel{N\to\infty}{\longrightarrow} 0. \tag{78}$$

Indeed, combining $f_a \le \prod_{e\in E} \xi_e^{m(a)}$ with the bound (64) and using the estimate (77) for h, the claim (78) follows. In view of (74), (75), and $\varphi_e\rho_{v_0,e_0} \in \Xi$, the second equality in the claim (71) follows for all initial weights a with sufficiently large $m(a)$. Using analytic continuation, it is also true for *all* $a \in (0,\infty)^E$. The claim (15) of Lemma 1 is equivalent to (16), which is a consequence of (71). ■

Proof of Theorem 1. Let $\pi = (v_0, \ldots, v_n)$ be a finite path in G, and set $\pi^j = (v_0, \ldots, v_j)$. Using the definition of the edge-reinforced random walk, we obtain

$$P_a[X_i = v_i \text{ for all } 0 \le i \le n] = \prod_{j=0}^{n-1} P_a[X_{j+1} = v_{j+1} \mid X_i = v_i \text{ for all } 0 \le i \le j]$$
$$= \prod_{j=0}^{n-1} \frac{(a + k(\pi^j))_{\{v_j,v_{j+1}\}}}{(I(a + k(\pi^j)))_{v_j}}. \tag{79}$$

For any $j \in \{0, \dots, n-1\}$, we apply equation (13) from Lemma 1 with $a + k(\pi^j)$ in place of a, v_j in place of v_0 and $\{v_j, v_{j+1}\}$ in place of $e_0 = \{v_0, v_1\}$. We write the last product in (79) as telescoping product:

$$\prod_{j=0}^{n-1} \frac{(a + k(\pi^j))_{\{v_j, v_{j+1}\}}}{(I(a + k(\pi^j)))_{v_j}} = \prod_{j=0}^{n-1} \frac{z(a + k(\pi^j) + k(v_j, v_{j+1}), v_{j+1})}{z(a + k(\pi^j), v_j)} = \frac{z(a + k(\pi), v_n)}{z(a, v_0)}; \tag{80}$$

in the last step we have used that $k(\pi^j) + k(v_j, v_{j+1}) = k(\pi^{j+1})$. This completes the proof of the first equality in (9) and thus the proof of the theorem. ■

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