Classical vs. Quantum Correlations

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During this session we will explore the significance of quantum information theory compared to classical information theory. The differences will be visualized with the help of a simple two-player game.

1 Classical Correlations

In this section the aim is to describe correlations using some “reasonable” assumptions. Later we will see that not all quantum correlations can be described using those assumptions.

1.1 LHV Ansatz

Assumption 1

We assume that quantities have values even before they are measured. They might be unknown, i.e. hidden, and thus a probabilistic description may be needed.

Assumption 2

The properties of one subsystem should not immediately depend on events taking place on very distant subsystems.

Example 1

Let us assume we have a source $S$ that emits pairs of particles to distant observers Alice $A$ and Bob $B$ who each perform $\pm 1$ valued measurements. Both can choose out of $m \in \mathbb{N}$ measurement devices, $A_x$ and $B_y$ with $x, y \in \{1, \ldots, m\}$:

$$
\begin{array}{c}
\pm 1 \rightarrow A_x \quad S \quad B_y \rightarrow \pm 1
\end{array}
$$
We denote by $p(a, b \mid x, y)$ the probability that $A$ measures outcome $a \in \{\pm 1\}$ while $B$ measures outcome $b \in \{\pm 1\}$ using devices $A_x$ and $B_y$, respectively. Here we want to consider the product of their expectations, i.e. $\langle A_x B_y \rangle := \sum_{a, b \in \{\pm 1\}} ab p(a, b \mid x, y)$.

**Ansatz 3 (LHV Ansatz)**

Let $A_x, B_y : \Omega \rightarrow \{\pm 1\}$ be random variables and $P$ a probability measure. We make the following ansatz for the expectation of the product:

$$\langle A_x B_y \rangle = \int_{\Omega} A_x(\omega) B_y(\omega) dP(\omega) \quad (1)$$

Here, $\omega \in \Omega$ plays the role of the hidden variable assumed in assumption 1 and locality from assumption 2 is expressed by the fact that $A_x$ and $B_y$ do not depend on $y$ and $x$, respectively. This ansatz is called the local hidden variable (LHV) ansatz.

### 1.2 Bell Inequalities

**Definition 1**

Let $C \in \mathbb{R}^{m \times m}$ with entries $C_{xy} := \langle A_x B_y \rangle$ that are empirically obtained expectation values for pairs of $\pm 1$ valued measurements. We denote by $\mathcal{C} \subseteq \mathbb{R}^{m \times m}$ the set of all such $C$ for which there exists an LHV description as in ansatz 3. An inequality $F : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ for $C$ is called Bell inequality if it holds for all $C \in \mathcal{C}$, i.e. if $F(C) \geq 0 \ \forall C \in \mathcal{C}$

**Remark 1**

1. $\mathcal{C}$ is a closed convex polytope.
2. $|C_{xy}| \leq 1$ are trivial Bell inequalities.

**Proposition 1**

Let $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ classically correlated (i.e. not entangled) and $M_x : \mathcal{B}(\{\pm 1\}) \rightarrow \mathcal{B}(\mathcal{H}_A)$, $M_y : \mathcal{B}(\{\pm 1\}) \rightarrow \mathcal{B}(\mathcal{H}_B)$ be POVMs for $x, y \in \{1, \ldots, m\}$. Let $C \in \mathbb{R}^{m \times m}$ be defined by

$$C_{xy} := \sum_{a, b \in \{\pm 1\}} ab \text{ tr} [\rho M_x(a) \otimes M'_y(b)] \quad (x, y = 1, \ldots, m), \quad (2)$$

then $C \in \mathcal{C}$.

**Proof.** If we define $A_x := \sum_{a \in \{\pm 1\}} a M_x(a)$, $B_y := \sum_{b \in \{\pm 1\}} b M'_y(b)$, then $-1 \leq A_x, B_y \leq 1$. 

2
Assume $\rho$ has convex combination $\rho = \sum_\omega p_\omega \rho_A(\omega) \otimes \rho_B(\omega)$. We calculate:

$$C_{xy} = tr[\rho A_x \otimes B_y] = \sum_\omega p_\omega tr\left[\rho_A(\omega) A_x\right] tr\left[\rho_B(\omega) B_y\right]$$

The last term gives a LHV description as in eq. (1) with $A_x(\omega) = tr\left[\rho_A(\omega) A_x\right]$ and $B_y(\omega) = tr\left[\rho_B(\omega) B_y\right]$ on some discrete probability space.

Since eq. (2) is just the expected value, the previous proposition yields that Bell inequalities hold in particular for any unentangled quantum state. Next we will state and prove a non-trivial Bell inequality.

**Theorem 2 (Causer-Horne-Shimony-Holt (CHSH) inequality)**

Every LHV theory for the description of $\pm 1$ valued measurements with measuring devices $A_1, A_2, B_1, B_2$ (as in ansatz 3) satisfies:

$$|\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| \leq 2 \quad (3)$$

**Proof.** By the definition of the expected values, we can rewrite the left-hand side as

$$\left| \int_\Omega A_1(\omega) (B_1(\omega) + B_2(\omega)) + A_2(\omega) (B_1(\omega) - B_2(\omega)) dP(\omega) \right|.$$

If we fix $\omega \in \Omega$, we can distinguish two cases:

1. $B_1(\omega) = B_2(\omega)$, i.e. $B_1(\omega) - B_2(\omega) = 0$ and $B_1(\omega) + B_2(\omega) \in \{\pm 2\}$.

2. $B_1(\omega) \neq B_2(\omega)$: Since there are only values $\pm 1$ possible, this implies $B_1(\omega) = -B_2(\omega)$, i.e. $B_1(\omega) + B_2(\omega) = 0$ and $B_1(\omega) - B_2(\omega) \in \{\pm 2\}$.

Since both $A_1(\omega)$ and $A_2(\omega)$ are in $\{\pm 1\}$, in both cases, the integrand is in $\{\pm 2\}$ which yields, using the fact that $P$ is a probability measure, the desired inequality. \qed

We will later see an example of how entangled states can violate this inequality.

## 2 The CHSH game

In this section we will first look at classical strategies for the CHSH game. Later we will also consider quantum strategies.
2.1 Definition

Definition 2 (Nonlocal Game with two Players)
A nonlocal game with two players (henceforth only nonlocal game) consists of players Alice, A, and Bob, B, and a referee Charlie, C. All communication in the game is only between any player and the referee, however the players are allowed to choose a strategy beforehand. The referee chooses a question at random for each of the two players. Every player then responds with an answer (dependent on the question she got). After receiving both answers, the referee determines whether both players won or lost the game. It is therefore not possible for one player to win and for the other one to lose. See also fig. 1.

Definition 3 (CHSH game)
The CHSH game is a nonlocal game with two players. The referee Charlie chooses questions $q_A, q_B$ for Alice and Bob, respectively, uniformly from the set $\{0, 1\}$. The answers of Alice and Bob, $r_A, r_B$, respectively, must be either 0 or 1. The players win if the following condition is met:

$$r_A \oplus r_B = q_A \land q_B.$$  \hspace{1cm} (4)

Here, $\oplus$ denotes the “exclusive or” which is equivalent to addition modulo 2. The following table shows the winning conditions:

<table>
<thead>
<tr>
<th>$(q_A, q_B)$</th>
<th>$r_A \oplus r_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>0</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>1</td>
</tr>
</tbody>
</table>
2.2 Classical Strategies

**Proposition 3**

*Using a classical strategy, i.e. Alice and Bob are only allowed to return classically correlated states, the maximal probability for Alice and Bob to win the game is \(3/4\), i.e. \(P(WIN) \leq 3/4\). This bound is sharp.*

**Proof.** For the sake of contradiction, let us assume that there is a strategy for Alice and Bob with \(P(WIN) > 3/4\). Since there are only 4 possible question-pairs, this means that \(P(WIN) = 1\), i.e. they would win every time. Let \(r_A, r_B : \{0, 1\} \to \{0, 1\}\) be the strategies used by Bob and Alice, accordingly. We write the winning conditions as a system of equations:

\[
\begin{align*}
    r_A(0) \oplus r_B(0) &= 0 \\
    r_A(0) \oplus r_B(1) &= 0 \\
    r_A(1) \oplus r_B(0) &= 0 \\
    r_A(1) \oplus r_B(1) &= 1
\end{align*}
\]

Adding those four equations modulo 2 gives \(0 = 1\), a contradiction. Even when using probabilistic strategies rather than deterministic ones, the expected probability to win will never be greater than with a deterministic one. So far we have proven that there cannot be a strategy \(P(WIN) > 3/4\), thus \(P(WIN) \leq 3/4\) must hold. Next we will give a strategy with \(P(WIN) = 3/4\): Suppose, Alice and Bob will always return 0 as an answer. Then in three out of four cases, they will win the game. Since the cases occur with equal probability (are chosen uniformly), this strategy wins with probability 3/4. \(\square\)

**Relation between the CHSH Game and the CHSH Inequality** We will look into the relation between the CHSH game and the CHSH inequality from the previous section. If we assume that Alice and Bob return values \(\{\pm 1\}\) rather than \(\{0, 1\}\), we get new winning conditions: Call the new answers \(\tilde{r}_A\) and \(\tilde{r}_B\) for Alice and Bob, respectively. In the case \((q_A, q_B) \neq (1, 1)\) the winning condition is that the sum of \(r_A\) and \(r_B\) is zero which is equivalent to saying that both are either 0 or 1. This is equivalent to saying that the product of \(\tilde{r}_A\) and \(\tilde{r}_B\) is one. Similarly, for \((q_A, q_B) = (1, 1)\), the winning condition is that \(r_A\) and \(r_B\) are different, i.e. the product of \(\tilde{r}_A\) and \(\tilde{r}_B\) is minus one. We summarize:
We now consider measuring devices $A_0, A_1$ for Alice and $B_0, B_1$ for Bob. Alice uses $A_0$ if $q_A = 0$ and $A_1$ if $q_A = 1$. Similarly for Bob. We are now in the situation of example 1, i.e. we can make a LHV ansatz. For $x, y \in \{0, 1\}$ consider the expectation of the product

$$
\langle A_x B_y \rangle = \sum_{a,b \in \{\pm 1\}} p(a, b \mid x, y) - \sum_{a,b \in \{\pm 1\}} p(a, b \mid a \neq b) \tag{5}
$$

For $(x, y) \neq (1, 1)$, eq. (5) gives the probability of winning minus the probability of losing, while for $(x, y) = (1, 1)$, eq. (5) gives the probability of losing minus the probability of winning. Since all devices are chosen with equal probability $1/4$, we get the total probability of winning minus the probability of losing by the following formula:

$$
P(WIN) - P(LOSE) = \frac{1}{4} [\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle] \tag{6}
$$

Using $P(WIN) = 1 - P(LOSE)$, we obtain $P(WIN) \leq 3/4$, the same bound as in proposition 3.

### 2.3 Quantum strategies

However, if Alice and Bob share an entangled two-qubit system which is initialized in the Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then they can gain a probability of $\cos^2(\pi/8) \approx 0.85$ if they use following strategy:

For a given angle $\theta \in [0, 2\pi)$, define:

$$
|\phi_0(\theta)\rangle = \cos(\theta) |0\rangle + \sin(\theta) |1\rangle \tag{8}
$$

$$
|\phi_1(\theta)\rangle = -\sin(\theta) |0\rangle + \cos(\theta) |1\rangle \tag{9}
$$

The strategy is now:
1. Alice takes the first qubit and Bob takes the second qubit from the quantum system.

2. If Alice receives $x = 0$ then she measures her qubit with respect to the basis
   \[
   \{|\phi_0(0)\rangle, |\phi_1(0)\rangle\}
   \]
   and if she receives the question 1, she will measure her qubit with respect to the basis
   \[
   \{|\phi_0(\pi/4)\rangle, |\phi_1(\pi/4)\rangle\}
   \]  

3. Bob uses a similar strategy, except that he measures with respect to the basis
   \[
   \{|\phi_0(\pi/8)\rangle, |\phi_1(\pi/8)\rangle\} \quad \text{or} \quad \{|\phi_0(-\pi/8)\rangle, |\phi_1(-\pi/8)\rangle\}
   \]
   depending on whether his question was 0 or 1.

4. Alice and Bob output the values obtained as their answers $a$ and $b$.

Each of these matrices is a rank-one projection matrix, so the measurements Alice and Bob are making are examples of projective measurements. Given our particular choice of $|\psi\rangle$, we have $\langle\psi| X \otimes Y |\psi\rangle = \frac{1}{2}tr(X^TY)$ for arbitrary matrices $X$ and $Y$. Thus, as each of the matrices $X^a$ and $Y^b$ is real and symmetric, the probability that Alice and Bob answer $(s, t)$ with $(a, b)$ is $\frac{1}{2}tr(X^a Y^b)$. It is now routine to check that in every case, the correct answer is given with probability $\cos^2(\pi/8)$ and the incorrect answer with probability $\sin^2(\pi/8)$.

3 Tseirelson bound

One might wonder if it is possible to do even better with another quantum strategy for the CHSH game. The answer is "no" due to Tseirelson’s bound:

**Theorem 4 (Tseirelson bound for the CHSH inequality)**

For $\hat{\beta} := A_1 \otimes (B_1 + B_2) + A_2 \otimes (B_1 - B_2)$ with $A_i, B_j$ as above and for all density operators $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$:

\[
\left| tr[\rho \hat{\beta}] \right| \leq 2\sqrt{2}
\]  

That is, quantum theory violates the CHSH inequality at most by a factor $\sqrt{2}$. Moreover, there exists a pure state on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and observables $A_x, A_y \in \mathcal{B}(\mathbb{C}^2)$ with eigenvalues $\pm 1$, s.t. equality holds in this equation.
Proof. Consider the map \( A_i \mapsto tr[\rho \hat{\beta}] \) for \( i \in \{1, 2\} \). Since these are affine functionals over the closed convex set \(-1 \leq A_i \leq 1\), the supremum and infimum is attained for some extreme point for which \( \text{spec}(A_i) = \{\pm 1\} \) and thus \( A_i^2 = 1 \). The same holds for \( B_j, j \in \{1, 2\} \). Using this property, direct computations leads to
\[
\hat{\beta}^2 = 4\mathbb{1} \otimes \mathbb{1} + [A_2, A_1] \otimes [B_1, B_2].
\]
(14)

We exploit this via positivity of the variance and obtain
\[
tr[\rho \hat{\beta}^2] \leq tr[\rho \hat{\beta}^2] = 4 + tr[\rho [A_2, A_1] \otimes [B_1, B_2]] \leq 4 + \| [A_2, A_1] \otimes [B_1, B_2] \| \leq 8
\]
(15)
where the last inequality uses that
\[
\| [A_1, A_2] \| \leq \| A_1 A_2 \| + \| A_2 A_1 \| \leq 2\| A_1 \| \| A_2 \| = 2
\]
and similarly for the B’s. So we end up with \( tr[\rho \hat{\beta}] \leq 2\sqrt{2} \) as claimed.

In order to prove that equality can be achieved, assume that \( \rho = |\psi\rangle \langle \psi| \) where \( |\psi\rangle \) is an eigenvector of \( \hat{\beta} \) with eigenvalue \( \nu \). Then equality holds in (15).

Now take \( A_1 = B_1 = \sigma_1 \) and \( A_2 = B_2 = \sigma_2 \) Pauli matrices, so that \( \hat{\beta}^2 = 4\mathbb{1} + 4\sigma_3 \otimes \sigma_3 \) has eigenvalues 0 and 8. Hence, \( \nu \) can be chosen such that \( tr[\rho \hat{\beta}]^2 = \nu^2 = 8 \).

Remark 2
We have previously seen that \( \hat{\beta} \) can be interpreted as the total probability of winning minus the probability of losing for the CHSH game. So for the previously used strategy we archive equality in Tsirelson’s bound: \( 4 \times (P(\text{WIN}) - P(\text{LOSE})) = 4 \times (\cos(\pi/8)^2 - \sin(\pi/8)^2) = 2\sqrt{2} \).

Remark 3
Note that if the local observables are commuting, i.e., \([A_2, A_1] = 0 \) or \([B_1, B_2] = 0 \), then we recover the classical bound \( \langle \text{CHSH} \rangle \leq 2 \). Clearly, the noncommutativity of observables in quantum mechanics plays a crucial role in the quantum advantage, yielding a winning probability of 85% compared to the classical 75% in the CHSH game.

Remark 4
The violation of CHSH by a factor of \( \sqrt{2} \) has been verified experimentally for the first time in the early 80’s. This was done using down conversion in a non-linear crystal, which produces entangled pairs of photons, whose polarization degrees of freedom violate CHSH. A more recent experiment [3] tested the CHSH Bell inequality on photon pairs in maximally entangled states of polarization in which a value \( 2.8276 \pm 0.00082 \) was observed.
The argumentation can be generalized to more than two observables: Consider $\langle A_x B_y \rangle := C_{x,y}$, $x, y \in \{1, \ldots, m\}$ as before, $\gamma \in \mathbb{R}^{m \times m}$ and define

$$
||\gamma||_{LHV} := \sup_{a, b \in \{\pm 1\}^m} \left| \sum_{x,y} \gamma_{xy} a_x b_y \right|
$$

(17)

$$
||\gamma||_{\text{quantum}} := \sup_{\rho, \{A_x B_y\}} \left| \sum_{x,y} \gamma_{xy} \text{tr}[\rho A_x \otimes B_y] \right|
$$

(18)

where $\rho$ is density operator and $-1 \leq A_x, B_y \leq 1$. The Trirelson bound for the CHSH inequality then reads:

$$
\nu(\gamma) := \frac{||\gamma||_{\text{quantum}}}{||\gamma||_{LHV}} = \sqrt{2} \text{ for } \gamma = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
$$

(19)

**Theorem 5 (General Cirelson bounds)**

1. $\gamma \in \mathbb{R}^{2 \times 2} \Rightarrow \nu(\gamma) \leq \sqrt{2}$

2. $\gamma \in \mathbb{R}^{m \times m}, m \in \mathbb{N} \Rightarrow \nu(\gamma) \leq K_G < \frac{\pi}{2m(1+\sqrt{2})} \approx 1.782$

**Remark 5**

1. means that the choice of coefficients in CHSH is optimal

2. is a non-trivial statement whose proof is based on a deep result of Grothendieck. $K_G$ is called Grothendieck’s constant, which is unknown but equal to the supremum of $\nu(\gamma)$ over all $\gamma \in \mathbb{R}^{m \times m}$ and all $m \in \mathbb{N}$

**References**

