Seminar Matrix Analysis
Summer Semester 2020

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Time & Location: beginning of term (block-seminar), TBD

Topics

We will discuss a selection of classical and more recent results of matrix analysis. Relevant topics include matrix norms, functional calculus, matrix monotone functions and convexity, matrix inequalities, majorization, matrix polynomial and polynomial identities, complete positivity, eigenvalues and Horn's conjecture.

We will mainly follow [2, 3, 6] and discuss the following topics. For each topic in the following list, the following results/definitions/examples should be discussed (at a minimum).

1. **Majorization:** [2, Chapter II]:
   - definition "majorized"/"(weakly) sub-majorized"
   - definition "doubly stochastic matrix"
   - possibly Exercises II.1.5 and/or II.1.7 in [2]
   - Thm. II.1.10(i) $\Leftrightarrow$ (iv)
   - Exercise II.1.12 as an example for majorization (and for unitary-/ortho-stochasticity), possibly Exercise II.1.13
   - Birkhoff's Theorem (Thm. II.2.3, Hall Theorem II.2.1)
   - Theorem II.3.1
   - possibly Equation (II.23)
2. **Matrix norms:** [2, Chapter IV].
   - norms on $\mathbb{C}^n$, including definition
   - connection to symmetric gauge functions
   - work out example IV.1.4
   - dual of a norm (Equation (IV.21)), show that this defines a norm, work out Exercise IV.1.13, Problem IV.5.4
   - work out Exercise IV.1.15 explicitly for $p = 1$ and for $p = 2$
   - definition "matrix norm", "sub-multiplicative norm", "unitarily invariant norm"
   - correspondence with symmetric gauge functions (Thm. IV.2.1, complete the proof)
   - operator norm, trace norm, Frobenius norm, Schatten p-norms, Ky Fan k-norms, induced norms (Problem IV.5.11)
   - sketch proof that unitarily invariant norms are sub-multiplicative, p. 94
   - Hilbert-Schmidt inner product (Equation IV.35), dual matrix norm (pg. 96)
   - weakly unitarily invariant norms, numerical radius
3. **Operator convex and operator monotone functions:** [2, Chapter 5]:
   - Löwner's partial order

- definition “operator monotonicity”, “operator convexity”
- examples in Section V.1, Proposition V.1.6, Theorem V.1.9
- Sketch proof idea and results from: Theorem V.2.3, Theorem V.2.5, Corollary V.2.6, Theorem V.2.9, Theorem V.2.10,
- Exercise V.2.11, Exercise V.2.13
- Löwner’s Theorems (Corollary V.4.5, Theorem V.4.6: outline basic ideas)

4. **Positive and completely positive maps:** [3, up to p. 71]
   - Definition of “positive (semi-)definite matrix”, “positive map”, “completely positive map”
   - One example of a positive map that is not completely positive.
   - The three basic representation theorems for completely positive maps: Kraus (Thm. 3.1.1), Stinespring dilation (Thm. 3.1.2), Choi (Thm. 3.1.4, especially (i) and (iii)).

5. **Matrix perturbation theory:** [6, Chapter 6], [2, Chapters VI-VIII] [7, Appendix C.3].
   - Gershgorin discs (Theorem 6.1.1 in [6] and Corollaries)
   - condition number, perturbation theorems 6.3.1 or 6.3.2 in [6]
   - spectral perturbation of normal matrices, e.g. Hoffman/Wielandt-Theorem (Theorem 6.3.5 in [6]; see also Thm VI.4.1 in [2]) or Weyl’s Perturbation Theorem (Thm. VI.2.1 in [2])
   - perturbation theory of non-degenerate eigenvalues and eigenvectors of non-normal matrices (6.3.10-6.3.12 in [6] with examples; see appendix C.3 in [7] for perturbation of eigenvectors)
   - Other facets:
     - spectral perturbation of normal matrices (section VI.3 in [2])
     - perturbation of eigenspaces of normal matrices (sections VII.3, VII.4 in [2])
     - general spectral variation bounds (VIII.1, VIII.2, VI.1 in [2])

6. **Matrix groups and semigroups:** [4], [8, Section 7.1.1], [5]
   - matrix groups [4], specializing to groups over the real or complex numbers, i.e. $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$
   - matrix groups over the complex/real numbers [4, section 1.1]
   - $GL(n)$, $SL(n)$, $O(n)$, $SO(n)$, $Sp(n)$, $U(n)$, $SU(n)$ [4, section 1.1]
   - preservation of bilinear forms, classical groups [4, section 1.8]
   - exponential map and tangent spaces (especially sections 2.2, second part of 2.3, 2.5, maybe 2.6, possibly 2.7 of [4])
   - one-parameter semigroups of matrices [8, Section 7.1.1] (see also [5] specializing to the finite-dimensional case): exposition in [8, Section 7.1.1], some examples adapted from [5]

7. **Matrix calculus** [2, Chapter X.4], [1, Chapters 1-2]
   - Frechet derivative
   - Gateaux derivative
   - rules of calculus
   - derivatives of some matrix functions
   - mean value theorem
   - higher derivatives
   - Taylor expansion

Guidelines

- The talks should be given in English.
- Presentations should be given predominantly on the blackboard but you may use a projector for showing, e.g., pictures or graphs. You may also use slides e.g., for reviewing materials as long as the total time doing so does not exceed 15–20 minutes.
- The target audience are the other students attending the seminar. Try to make sure it is understandable for everyone.
- You are encouraged to submit a summary (at most 2 pages) of the assigned topic, preferably at least one week before your talk.
- You can arrange a meeting with me or one of my assistants prior to the presentation. This is intended to help identify key concepts to be presented, address specific technical questions, and to make sure your presentation is ready for public consumption. These meetings should organized topic-wise.
- Attendance of and active participation in all talks except in justified circumstances will be required to receive credits.

Other information

This seminar will be held as a block-seminar at the beginning of term, most likely at the end of April (week 2 of lectures). Presentations should therefore be prepared during the semester break. The exact date/times will be decided once the number of participants is determined.

References

[1] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge University Press, 1993

[2] R. Bhatia, *Matrix Analysis*, Springer Graduate Texts in Mathematics, 1997. Available here (see proxy instructions below).

[3] R. Bhatia, *Positive Definite Matrices*, Princeton University Press, 2009, available here via OPAC.

[4] M. Boij and D. Laksov, *An Introduction to Algebra and Geometry via Matrix Groups*, lecture notes 1995, available here.

[5] K.-J. Engel and R. Nagel, *A short course on operator semigroups*, Springer 2000. Available here

[6] A. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, 2nd Edition, 2013.

[7] D. MacKay, *Information Theory, Inference, and Learning Algorithms*, Cambridge University Press, 2003. Available here.

[8] M. Wolf, *Quantum Channels and Operations*, Lecture Notes, 2012, available here.

Note: You may have to follow the Proxy setup instructions to access some of these materials through the university's network.