Output entropy of tensor products of random quantum channels [1]

Motohisa Fukuda

[1] [Collins, Fukuda and Nechita], [Collins, Fukuda and Nechita]

Aim

1. Let Φ be a random quantum channel:

$$\Phi : \mathcal{M}_{N_n}(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$$

   with its environment space being $\mathbb{C}^k$.

2. Then, we ask: what is the asymptotic behavior of output eigenvalues of

$$\Phi \otimes \bar{\Phi}(|a_n\rangle\langle a_n|)$$

   as n and N_n grow? [2] Here, $\bar{\Phi}$ is the complex conjugate of Φ.

3. To make this question tractable, we need to define a good measure on quantum channels and impose some conditions on the sequence of $|a_n\rangle \in \mathbb{C}_{N_n}$.

[2] The case where $|a_n\rangle$ is a Bell state is done in [Collins and Nechita]

Background

1. It is shown in [Hastings] that for some channels Φ

$$S_{\min}(\Phi \otimes \bar{\Phi}) < S_{\min}(\Phi) + S_{\min}(\bar{\Phi})$$

Here,

$$S_{\min}(\Phi) = \min_{\rho} S(\Phi(\rho))$$

where $S(\cdot)$ be the von Neumann entropy.

2. This phenomena, called *additivity violation*, solved other important problems in quantum information theory via the equivalence proven in [Shor].

3. An important factor in the proof on additivity violation is that

$$\Phi \otimes \bar{\Phi}(|b\rangle\langle b|)$$

has some eigenvalue more than N/kn when $|b\rangle$ is a Bell state, proven in [Hayden and Winter].

Hayden - Winter trick

1. Let's prove the property for random unitary channels:

$$\Psi(\rho) = \sum_{i=1}^{k} p_i U_i \rho U_i^*$$

   where $U_i \in \mathcal{U}(n)$ are unitary matrices and $\{p_i\}$ is a probability distribution.

2. Remember that, for a Bell state $|b\rangle$, we have

$$U \otimes \bar{U} |b\rangle = |b\rangle$$

3. Now, with the fact $\sum_{i=1}^{k} p_i^2 \geq \frac{1}{k}$

$$\Psi \otimes \bar{\Psi}(|b\rangle\langle b|) = \sum_{i=1} p_i^2 |b\rangle\langle b| + \sum_{i \neq j} p_i p_j (U_i \otimes \bar{U}_j)|b\rangle\langle b|(U_i^* \otimes \bar{U}_j^T)$$

   prove the statement because $N = n$.

Conditions on input sequences [3]

1. Assumption 0:
   Of course, in order to satisfy unit trace condition, we have
   $$\mathrm{Tr}[A_n A_n^*] = 1$$
2. Assumption 1:
   $$\frac{\mathrm{Tr}\left[A_n\right]}{\sqrt{N_n}} = m + O\left(\frac{1}{n^2}\right)$$
   where m is an important parameter.
3. Assumption 2:
   $$\|A_n\|_\infty = O\left(\frac{1}{\sqrt{n}}\right)$$
   where A_n must be stable for the distribution to have the limit.

[3] We have the identification $a_n = A_n$ where $A_n \in \mathcal{M}_{N_n}(\mathbb{C})$

Examples of input sequences

1. Example 1: Bell state

$$|a_n\rangle = \frac{1}{\sqrt{N_n}} \sum_{j=1}^{N_n} |j\rangle|j\rangle$$

$$A_n = \frac{1}{\sqrt{N_n}} \cdot \mathrm{diag}\{1, \ldots, 1\}$$

$$m = 1$$

2. Example 2: Bell state with phase

$$|a_n\rangle = \frac{1}{\sqrt{N_n}} \sum_{j=1}^{N_n} \exp\left\{\frac{2\pi i}{N_n} j\right\} |j\rangle|j\rangle$$

$$A_n = \frac{1}{\sqrt{N_n}} \mathrm{diag}\left\{\exp\left\{\frac{2\pi i}{N_n}\right\}, \exp\left\{\frac{4\pi i}{N_n}\right\}, \ldots, 1\right\}$$

$$m = 0$$

Setup for random quantum channels

1. A quantum channel $\Phi : \mathcal{M}_{N_n} \to \mathcal{M}_n$ with its environment space being of dimension k can be written as

$$\Phi(\rho) = \mathrm{Tr}_{\mathbb{C}^k}\left[V\rho V^*\right]$$

for some isometry $V : \mathbb{C}^{N_n} \to \mathbb{C}^{kn}$.

2. Then, we define a measure on quantum channels with respect to the Haar measure on $\mathcal{U}(kn)$, i.e., we can get the random isometry V by truncating $U \in \mathcal{U}(kn)$
3. Its complex conjugate is defined as

$$\bar{\Phi}(\rho) = \mathrm{Tr}_{\mathbb{C}^k}\left[\bar{V}\rho V^T\right]$$

Our random matrix

1. Our random matrix $Z_n \in \mathbb{M}_{n^2}(\mathbb{C})$ to be analyzed is :

$$Z_n = \Phi \otimes \bar{\Phi}(|a_n\rangle\langle a_n|)$$

2. We set $N = tkn$ with $0 < t < 1$ where k is fixed.
3. We investigate asymptotic behavior of the empirical eigenvalue distribution of a matrix, which is the probability measure:

$$\frac{1}{k^2}\sum_{i=1}^{k^2} \delta_{\lambda_i},$$

where $\lambda_1, \ldots, \lambda_{k^2}$ are (asymptotically) the non-zero eigenvalues of Z_n.

Limiting measure

1. Under the assumptions on input sequences, the empirical eigenvalue distribution of the matrix Z_n converges *almost surely*, as $n \to \infty$, to the probability measure

$$\frac{1}{k^2}\left[\delta_{\lambda_1} + (k^2 - 1)\delta_{\lambda_2}\right]$$

where the Dirac masses are located at

$$\lambda_1 = t|m|^2 + \frac{1 - t|m|^2}{k^2} \quad \text{and} \qquad \lambda_2 = \frac{1 - t|m|^2}{k^2}.$$

2. In other words, the output state has asymptotically the following eigenvalues:
   - $t|m|^2 + \frac{1-t|m|^2}{k^2}$, with multiplicity one;
   - $\frac{1-t|m|^2}{k^2}$, with multiplicity $k^2 - 1$.

Analysis

1. Among inputs under the assumptions a Bell input gives the least entropy for random *conjugate pair* $\Phi \otimes \bar{\Phi}$.
2. It is because the entropy is monotone in $|m|$ but on the other hand, $\mathrm{Tr}[AA^*] = 1$ implies that

$$\left| \frac{\mathrm{Tr}\,[A_n]}{\sqrt{N_n}} \right| \le 1 \qquad \text{i.e.} \qquad |m| \le 1$$

   The inequality is attained when $A_n = I/\sqrt{N_n}$ up to a global complex phase.
3. This is a positive evidence supporting the conjecture that Bell states give the least entropy for generic conjugate pairs $\Phi \otimes \bar{\Phi}$. [4]

[4] I.e., we ultimately want to fix $\Phi \otimes \bar{\Phi}$ and then know the optimizers.

Setup for "random random" unitary channels

1. A random unitary channel $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ is defined by

$$\Psi(\rho) = \sum_{i=1}^{k} w_i U_i \rho U_i^*$$

   where $U_i \in \mathcal{U}(n)$ and $(w_i)_i$ is a probability distribution.
2. If we take $(U_i)_i$ to be i.i.d. with respect to the Haar measure on $\mathcal{U}(n)$, we can define "random random" unitary channels.
3. Its complex conjugate is defined by

$$\bar{\Psi}(\rho) = \sum_{i=1}^{k} w_i \bar{U}_i \rho U_i^T$$

Result for random unitary channels

1. Similarly as before, we are interested in the following random matrix $Z_n \in \mathbb{M}_{n^2}(\mathbb{C})$:

$$Z_n = \Psi \otimes \bar{\Psi}(|a_n\rangle\langle a_n|)$$

2. For simplicity, we set $w_i = 1/k$ for all $1 \leq i \leq k$ in this slide. [5]
3. Then, assuming the same conditions on the sequence $(a_n)_n$, the limit eigenvalue distribution turns out to be

$$\left(\frac{|m|^2}{k} + \frac{1-|m|^2}{k^2}, \underbrace{\frac{1-|m|^2}{k^2} \cdots \frac{1-|m|^2}{k^2}}_{k-1 \text{ times}}, \underbrace{\frac{1}{k^2}, \ldots, \frac{1}{k^2}}_{k^2-k \text{ times}} \right)$$

[5] In our paper, $(w_i)_i$ is another parameters.