Partial transpose of random quantum states and meander polynomials [1]

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[1] [Fukuda and Śniady]

States and their partial transpose

1. A (quantum) state ρ in finite dimension is a positive Hermitian matrix of trace one on a Hilbert space $\mathbb{C}^d$. We take $d = mn$ so that we consider bipartite quantum states on $\mathbb{C}^m \otimes \mathbb{C}^n$.
2. Write ρ in the canonical basis $\{|i\rangle\}$ of the first space as:

$$\rho = \sum |i\rangle\langle j| \otimes \rho_{i,j}$$

then transpose on the first space defines its partial transpose:

$$\rho^\Gamma = \sum |j\rangle\langle i| \otimes \rho_{i,j}$$

3. A state ρ and its transpose ρ^T share the same eigenvalues, so we don't really care to which space transpose is applied.

Separability

1. A quantum state ρ is called *separable* if it is written as a convex combination of products states:

$$\rho = \sum_i \sigma_i^{(1)} \otimes \sigma_i^{(2)}$$

where $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$ are positive Hermitian matrices.

2. Otherwise, a quantum state is called *entangled*.
3. For a separable state, its partial transpose is always positive:

$$\rho = \sum_i \left(\sigma_i^{(1)}\right)^T \otimes \sigma_i^{(2)}$$

Hence,

- If the partial transpose is negative, the quantum state is entangled [Peres].
- The converse is not generally true except for $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ [Horodecki, Horodecki, Horodecki].

Random states

1. This is why, it is interesting to ask whether partial transpose of random states typically have negative eigenvalues.
2. We investigate more generally asymptotic eigenvalue distribution of partial transpose of random quantum states.
3. We define random quantum state as follows. [2]

$$\underbrace{|\phi\rangle\langle\phi| \in \mathcal{M}_{lmn}(\mathbb{C})}_{\text{random pure states}} \mapsto \underbrace{\mathrm{Tr}_{\mathbb{C}^l}[|\phi\rangle\langle\phi|] \in \mathcal{M}_{mn}(\mathbb{C})}_{\text{random mixed states}}$$

What we do:

- Take the uniform random rank-one projections on $\mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^n$.
- Then, apply partial trace to the space of $\mathbb{C}^l$ to get the random reduced states.

[2] To know about this push-forward measure, see [Życzkowski and Sommers].

Moment method

1. We calculate the moments of random matrix ρ^{Γ} by an exact Wingarten Calculus formula. [3]
2. Taking the limit in the dimension, the polynomial formula of the pth-moment gives geodesics in the permutation group S_p
3. Some definitions:
   - For permutations $\alpha, \beta \in S_p$, define the distance as

$$\mathrm{dist}(\alpha, \beta) = p - \#(\alpha^{-1}\beta)$$

     where $\#\alpha$ is the number of cycles in α.
   - For given permutations $\alpha, \gamma \in S_p$ we write

$$\alpha \to \beta \to \gamma$$

     if $\beta \in S_p$ is on a geodesic between α and γ.

[3] For the exact formula, see [Collins and Śniady].

Three regimes we know to be interesting

1. $l \sim mn$:

$$\mathrm{id} \to \alpha \to \pi \quad \text{and} \quad \mathrm{id} \to \alpha \to \pi^{-1}$$

2. m is fixed and $l \sim n$:

$$\mathrm{id} \to \alpha \to \pi$$

3. l is fixed and $m \sim n$:

$$\pi^{-1} \to \alpha \to \pi$$

Here, $\pi = (1, 2, \ldots, p)$ is the canonical maximal cycle.

Aubrun's regime[4]: $l \sim mn$

1. Let $\frac{mn}{l} = a > 0$ and then the empirical eigenvalue distribution has the following limit measure, as a semi-circle distribution:

$$\frac{d\mathrm{SC}_{1,\sqrt{a}}}{dx} = \frac{1}{2\pi a}\sqrt{4a - (x-1)^2} \qquad \text{for } |x-1| \leq 2\sqrt{a}.$$

2. This measure has a compact support $[1-2\sqrt{a}, 1+2\sqrt{a}]$. So, $a = \frac{1}{4}$ is the threshold for PPT or non-PPT.
3. The edge convergence of the distribution is proven. To do so,
   - Aubrun worked on the moments of

$$\rho^{\Gamma} - \mathrm{diag}(\rho^{\Gamma})$$

   - We [5] considered those of

$$\rho^{\Gamma} - \tilde{I}$$

   which gives more problems of more combinatorial nature.

[4] [Aubrun]

[5] [Fukuda and Śniady]

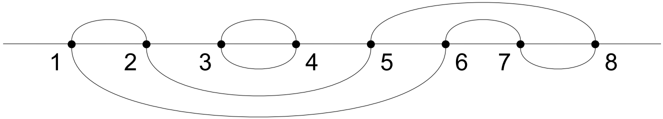
Other two regimes

1. **Banica-Nechita's regime[6]:** m is fixed and $l \sim n$
   - The associated geodesic is
   $$\mathrm{id} \to \alpha \to \pi$$
   - The limit distribution is *free difference of free Poisson laws*.
2. **Our new regime[7]:** l is fixed and $m = n$
   - The geodesic is
   $$\pi^{-1} \to \alpha \to \pi$$
   - The limiting moments yield the meander polynomials.

[6] [Banica and Nechita]
[7] [Fukuda and Śniady]

Meanders [8]

1. Suppose we have an infinite straight river and $2n$ bridges over the river. Then, a meander is a collection of closed self-avoiding and non-crossing connected roads passing through all of the bridges
2. In other words, a meander of order n consists of loops crossing a straight line at $2n$ points.
3. For example when $n = 4$ one of possible loops is:



[8] See, for example, [Di Francesco, Golinelli and Guitter]

Meanders with permutations

1. If one crosses a bridge then must cross another. For each side of the river, all the choices correspond to the non-crossing pair partitions of $2n$ elements, denoted by $\mathrm{NC}_2(2n)$.
2. Our example can be interpreted as (σ_1, σ_2) where
$\sigma_1 = \big\{\{1,2\},\{3,4\},\{5,8\},\{6,7\}\big\}$;
$\sigma_2 = \big\{\{1,6\},\{2,5\},\{3,4\},\{7,8\}\big\}$.
3. For each $k = 1, 2, \ldots n$, we define $M_n^{(k)}$ to be the number of meanders with k connected components. Then, the meander polynomial $M_n(x)$ is defined to be

$$M_n(x) = \sum_{k=1}^{n} x^k M_n^{(k)}.$$

The limiting moments - when l is fixed and $m = n$.

1. The limiting moment after rescaling survives only when p is even:

$$\sum_{\pi^{-1}\to\alpha\to\pi} l^{\#\alpha} = \sum_{\mathrm{id}\to\tau\to\pi^2} l^{\#(\pi^{-1}\tau)}$$

where, interestingly,

$$\pi^2 = (1, 3, \dots, p-1)(2, 4, \dots, p)$$

2. Therefore the above formula is rewritten as

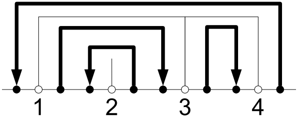
$$\sum_{\tau_1,\tau_2} l^{\#[\pi^{-1}(\tau_1\oplus\tau_2)]}.$$

where $\tau_1 \in \mathrm{NC}(\{1, 3, \dots, p-1\})$ and $\tau_2 \in \mathrm{NC}(\{2, 4, \dots, p\})$.

3. Here, $NC(\cdot)$ stands for *non-crossing partitions* of $p/2$ elements and it is embedded in to $S_{p/2}$ with respect to π.

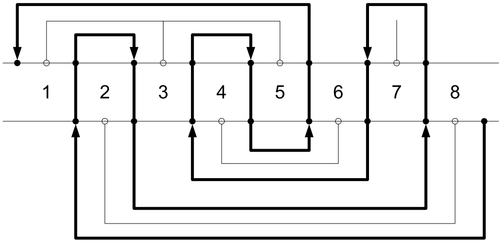
Recap.

1. There is a bijective correspondence between $\mathrm{NC}(q)$ and $\mathrm{NC}_2(2q)$, commonly known as *fattening*:
2. Suppose we are given $\alpha \in \mathrm{NC}(q)$.
   - We add two points i_- and i_+ for both sides of each $i \in \{1, \ldots, q\}$, left and right respectively.
   - we connect i_+ and j_- if $\alpha(i) = j$.
3. Example for $\alpha = \{1, 3, 4\}\{2\}$. We also use arrows to show the action of the permutation α.



Unveiling the connection - an example

1. Draw $\tau_1 = (1,3,5)(7)$ over the two parallel lines and $\tau_2 = (2,8)(4,6)$ under, in the picture below.
2. The fattenings are drawn by the thick lines.
3. As a whole, $\#[\pi^{-1}(\tau_1 \oplus \tau_2)]$ turns out to be a maximal loop, identifying 1_- and 8_+.



Unveiling the connection - the general procedure

Calculation of $\#[\pi^{-1}(\tau_1 \oplus \tau_2)]$ is systematized as:

1. Draw two horizontal lines with odd-numbered points on the upper line and even-numbered points on the lower line.
2. Draw the graphical representation for τ_1 above the upper line and that for τ_2 below the lower line.
3. Identify $(2i+1)_-$ and $(2i)_+$, and then $(2i)_-$ and $(2i-1)_+$.

This shows the following one-to-one correspondence.

$$NC(q) \times NC(q) \leftrightarrow NC_2(2q) \times NC_2(2q) \leftrightarrow \text{meanders of order } q$$

Unveiling the connection - the final step

Therefore, for each $q \in \mathbb{N}$,

$$\sum_{\tau_1,\tau_2 \in \mathrm{NC}(q)} l^{\#[\pi^{-1}(\tau_1 \oplus \tau_2)]} = \sum_{\tau_1,\tau_2 \in \mathrm{NC}(q)} l^{\#\text{ of loops in the "meander view"}}$$

$$= \sum_{k=1}^{q} l^k M_q^{(k)} = M_q(l),$$

which is the meander polynomial with order q and variable l.