Additivity questions of quantum channels

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Quantum states and channels

1. A quantum state ρ (in finite dimension) is a positive semi-definite Hermitian operator of trace one on a Hilbert space $\mathbb{C}^n$.
2. A physical picture of (quantum) channels with a k-dimensional environment is [1]

$$\Phi(\rho) = \mathrm{Tr}_{\mathbb{C}^k}\left[U(ee^* \otimes \rho)U^*\right]$$

Here, $e \in \mathbb{C}^k$ is a fixed unit vector in the environment and $U \in \mathcal{U}(kn)$ is a unitary matrix.

[1] $e = |e\rangle$ and $e^* = \langle e|$ in the conventional bra-ket notation.

Complementary channels

1. When the input $\rho = xx^*$ is a rank-one projection the following two matrices share the same non-zero eigenvalues.

$$\mathrm{Tr}_{\mathbb{C}^k}\left[U(ee^* \otimes xx^*)U^*\right] \sim \mathrm{Tr}_{\mathbb{C}^n}\left[U(ee^* \otimes xx^*)U^*\right]$$

2. Indeed, $U(e \otimes x) \in \mathbb{C}^k \otimes \mathbb{C}^n$ has the Schmidt decomposition:

$$\sum_i r_i |u_i\rangle \otimes |v_i\rangle$$

where $r_i > 0$, and $\{u_i\}, \{v_i\}$ are orthonormal in $\mathbb{C}^k$ and $\mathbb{C}^n$.

3. We define the complementary channel of Φ by [2]

$$\Phi^C(\rho) = \mathrm{Tr}_{\mathbb{C}^n}\left[U(ee^* \otimes \rho)U^*\right]$$

[2] [Holevo], [King, Matsumoto, Nathanson, Ruskai].

Minimum output entropy (MOE)

1. The von Neumann entropy $S(\cdot)$ of quantum state ρ is:

$$S(\rho) = -\mathrm{Tr}[\rho \log \rho] = -\sum_{i=1}^{d} \lambda_i \log \lambda_i$$

where λ_i are eigenvalues of ρ. Note that

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$$

2. The minimal output entropy of channel Φ is defined by

$$S_{\min}(\Phi) = \min_{\rho} S(\Phi(\rho))$$

where ρ are input states. [3]

[3] [King and Ruskai]

Holevo capacity (HC)

1. Holevo capacity of channel Φ is defined as: [4]

$$\chi(\Phi) = \max_{p_i,\rho_i} S(\Phi(\sum_i p_i\rho_i)) - \sum_i p_i S(\Phi(\rho_i))$$

   where $\{p_i, \rho_i\}$ is an ensemble.

2. We have an easy bound:

$$\chi(\Phi) \leq \log d - S_{\min}(\Phi)$$

3. The above bound is satiated when, for example,

$$\Phi(U_g \rho U_g^*) = U_g \Phi(\rho) U_g^*$$

   where $g \mapsto U_g$ is an irreducible adjoint representation. [5]

[4] [Holevo], [Schumacher and Westmoreland]
[5] [Holevo]

Remarks on MOE and HC

1. MOE measures purity of channels by considering optimal output while HC is connected to the capacity $C(\cdot)$:

$$C(\Phi) = \lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n})$$

2. Since von Neumann entropy is concave, MOE is achieved by pure input states. So,

$$S_{\min}(\Phi) = S_{\min}(\Phi^C)$$

3. To calculate HC, we need to know the geometry of output states, and in general

$$\chi(\Phi) \neq \chi(\Phi^C)$$

Additivity violation

1. Write quantum channels:

$$\Phi(\rho) = \mathrm{Tr}_{\mathbb{C}^k}\left[U(ee^* \otimes \rho)U^*\right]$$

   and their complex conjugate channels:

$$\bar{\Phi}(\rho) = \mathrm{Tr}_{\mathbb{C}^k}\left[\bar{U}(ee^* \otimes \rho)U^T\right]$$

2. Then, we have additivity violation [6] ; for some channels Φ,

$$S_{\min}(\Phi \otimes \bar{\Phi}) < S_{\min}(\Phi) + S_{\min}(\bar{\Phi})$$

3. Note that, for any channels Φ and Ω,

$$\min_{\rho \otimes \sigma} S((\Phi \otimes \Omega)(\rho \otimes \sigma)) = \min_{\rho} S(\Phi(\rho)) + \min_{\sigma} S(\Omega(\sigma))$$

[6] [Hastings]: more precisely, random unitary channels are used.

Entangled inputs can improve the capacity - sketchy

1. We know that there is a channel such that

$$S_{\min}(\Phi \otimes \bar{\Phi}) = S_{\min}(\Phi) + S_{\min}(\bar{\Phi}) - \epsilon \qquad \text{where} \quad \epsilon > 0$$

2. This implies [7] that $\Omega = \Phi \oplus \bar{\Phi}$ gives

$$S_{\min}(\Omega^{\otimes 2}) = 2S_{\min}(\Omega) - \epsilon$$

Then, there exists [8] a channel Ψ such that

$$\chi(\Psi \otimes \Psi) \geq 2\chi(\Psi) + \epsilon$$

where $\chi(\cdot)$ is the Holevo capacity.

3. So, we have

$$C(\Psi) = \lim_{n\to\infty} \frac{1}{2n} \cdot \chi\left(\Psi^{\otimes 2n}\right) \geq \lim_{n\to\infty} \frac{1}{2} \cdot \chi\left(\Psi^{\otimes 2}\right) = \chi(\Psi) + \frac{\epsilon}{2}$$

Entangled inputs improve the classical capacity: $C(\cdot)$.

[7] [Fukuda and Wolf]
[8] [Shor]

Additivity question for regularized quantities

1. Classical capacity:

$$C(\Phi \otimes \Omega) \stackrel{?}{=} C(\Phi) + C(\Omega) \qquad \text{for } \forall \Phi \neq \Omega$$

2. Regularized minimum output entropy:

$$\bar{S}_{\min}(\Phi \otimes \Omega) \stackrel{?}{=} \bar{S}_{\min}(\Phi) + \bar{S}_{\min}(\Omega) \qquad \text{for } \forall \Phi \neq \Omega$$

Here,

$$\bar{S}_{\min}(\Phi) = \lim_{N \to \infty} \frac{1}{N} \cdot S_{\min}(\Phi^{\otimes N})$$

3. Non-additivity of $\bar{S}_{\min}(\cdot)$ implies non-additivity of $C(\cdot)$. [9]

[9] See, for example, [Fukuda]

Finding counterexamples

1. Can we find concrete examples for additivity violation?
2. Concrete counterexamples for the following additivity violation were found: [10]

$$S_{p,\min}(\Phi \otimes \Phi) < S_{p,\min}(\Phi) + S_{p,\min}(\Phi) \qquad p > 2$$

Here,

$$S_{p,\min}(\Phi) = \min_{\rho} S_p(\Phi(\rho))$$

where S_p is the Renyi p-entropy.

3. Concrete counterexamples for $1 \leq p \leq 2$ are still open.

[10] [Grudka, M. Horodecki,Pankowski]

Important examples for additivity

1. Entanglement-breaking channels: [Shor][King]
2. Depolarizing channels: [King]
3. Unital qubit channels: [King]

For the above channels Φ, we know that, with any channel Ω,

$$S_{\min}(\Phi \otimes \Omega) = S_{\min}(\Phi) + S_{\min}(\Omega)$$
$$\chi(\Phi \otimes \Omega) = \chi(\Phi) + \chi(\Omega)$$

In particular, with any channel Ω,

$$\bar{S}_{\min}(\Phi \otimes \Omega) = \bar{S}_{\min}(\Phi) + \bar{S}_{\min}(\Omega)$$
$$C(\Phi \otimes \Omega) = C(\Phi) + C(\Omega)$$