Additivity questions of quantum channels

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Quantum states and channels

1. A quantum state $\rho$ (in finite dimension) is a positive semi-definite Hermitian operator of trace one on a Hilbert space $\mathbb{C}^n$.

2. A physical picture of (quantum) channels with a $k$-dimensional environment is

$$\Phi(\rho) = \text{Tr}_{\mathbb{C}^k} [U(ee^* \otimes \rho)U^*]$$

Here, $e \in \mathbb{C}^k$ is a fixed unit vector in the environment and $U \in \mathcal{U}(kn)$ is a unitary matrix.

\[\text{\textsuperscript{1}e = |e\rangle and } e^* = \langle e| \text{ in the conventional bra-ket notation.}\]
Complementary channels

1. When the input $\rho = xx^*$ is a rank-one projection the following two matrices share the same non-zero eigenvalues.

$$\text{Tr}_{\mathbb{C}^k} [U(ee^* \otimes xx^*)U^*] \sim \text{Tr}_{\mathbb{C}^n} [U(ee^* \otimes xx^*)U^*]$$

2. Indeed, $U(e \otimes x) \in \mathbb{C}^k \otimes \mathbb{C}^n$ has the Schmidt decomposition:

$$\sum_i r_i |u_i\rangle \otimes |v_i\rangle$$

where $r_i > 0$, and $\{u_i\}, \{v_i\}$ are orthonormal in $\mathbb{C}^k$ and $\mathbb{C}^n$.

3. We define the complementary channel of $\Phi$ by \(^2^2\)

$$\Phi^C(\rho) = \text{Tr}_{\mathbb{C}^n} [U(ee^* \otimes \rho)U^*]$$

\(^2^2\)[Holevo], [King, Matsumoto, Nathanson, Ruskai].
Minimum output entropy (MOE)

1. The von Neumann entropy $S(\cdot)$ of quantum state $\rho$ is:

$$S(\rho) = -\text{Tr}[\rho \log \rho] = - \sum_{i=1}^{d} \lambda_i \log \lambda_i$$

where $\lambda_i$ are eigenvalues of $\rho$. Note that

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$$

2. The minimal output entropy of channel $\Phi$ is defined by

$$S_{\text{min}}(\Phi) = \min_{\rho} S(\Phi(\rho))$$

where $\rho$ are input states.  

$^{3}$[King and Ruskai]
Holevo capacity (HC)

1. Holevo capacity of channel $\Phi$ is defined as: $^4$

$$\chi(\Phi) = \max_{\{p_i, \rho_i\}} S(\Phi(\sum_i p_i \rho_i)) - \sum_i p_i S(\Phi(\rho_i))$$

where $\{p_i, \rho_i\}$ is an ensemble.

2. We have an easy bound:

$$\chi(\Phi) \leq \log d - S_{\text{min}}(\Phi)$$

3. The above bound is satiated when, for example,

$$\Phi(U_g \rho U_g^*) = U_g \Phi(\rho) U_g^*$$

where $g \mapsto U_g$ is an irreducible adjoint representation. $^5$

$^4$[Holevo], [Schumacher and Westmoreland]

$^5$[Holevo]
Remarks on MOE and HC

1. MOE measures purity of channels by considering optimal output while HC is connected to the capacity $C(\cdot)$:

$$C(\Phi) = \lim_{n \to \infty} \frac{1}{n} \chi(\Phi \otimes^n)$$

2. Since von Neumann entropy is concave, MOE is achieved by pure input states. So,

$$S_{\text{min}}(\Phi) = S_{\text{min}}(\Phi^C)$$

3. To calculate HC, we need to know the geometry of output states, and in general

$$\chi(\Phi) \neq \chi(\Phi^C)$$
Additivity violation

1 Write quantum channels:

\[ \Phi(\rho) = \text{Tr}_{C^k} [U(ee^* \otimes \rho)U^*] \]

and their complex conjugate channels:

\[ \bar{\Phi}(\rho) = \text{Tr}_{C^k} \left[ \bar{U}(ee^* \otimes \rho)U^T \right] \]

2 Then, we have additivity violation \(^6\); for some channels \( \Phi \),

\[ S_{\text{min}}(\Phi \otimes \bar{\Phi}) < S_{\text{min}}(\Phi) + S_{\text{min}}(\bar{\Phi}) \]

3 Note that, for any channels \( \Phi \) and \( \Omega \),

\[ \min_{\rho \otimes \sigma} S((\Phi \otimes \Omega)(\rho \otimes \sigma)) = \min_{\rho} S(\Phi(\rho)) + \min_{\sigma} S(\Omega(\sigma)) \]

\(^6\) [Hastings]: more precisely, random unitary channels are used.
Entangled inputs can improve the capacity - sketchy

1. We know that there is a channel such that

\[ S_{\min}(\Phi \otimes \bar{\Phi}) = S_{\min}(\Phi) + S_{\min}(\bar{\Phi}) - \epsilon \]

where \( \epsilon > 0 \)

2. This implies that \( \Omega = \Phi \oplus \bar{\Phi} \) gives

\[ S_{\min}(\Omega \otimes^2) = 2S_{\min}(\Omega) - \epsilon \]

Then, there exists a channel \( \Psi \) such that

\[ \chi(\Psi \otimes \Psi) \geq 2\chi(\Psi) + \epsilon \]

where \( \chi(\cdot) \) is the Holevo capacity.

3. So, we have

\[
C(\Psi) = \lim_{n \to \infty} \frac{1}{2n} \cdot \chi(\Psi \otimes 2^n) \geq \lim_{n \to \infty} \frac{1}{2} \cdot \chi(\Psi \otimes^2) = \chi(\Psi) + \frac{\epsilon}{2}
\]

Entangled inputs improve the classical capacity: \( C(\cdot) \).

\(^7\)Fukuda and Wolf
\(^8\)Shor
Additivity question for regularized quantities

1. Classical capacity:

\[ C(\Phi \otimes \Omega) \overset{?}{=} C(\Phi) + C(\Omega) \quad \text{for } \forall \Phi \neq \Omega \]

2. Regularized minimum output entropy:

\[ \bar{S}_{\text{min}}(\Phi \otimes \Omega) \overset{?}{=} \bar{S}_{\text{min}}(\Phi) + \bar{S}_{\text{min}}(\Omega) \quad \text{for } \forall \Phi \neq \Omega \]

Here,

\[ \bar{S}_{\text{min}}(\Phi) = \lim_{N \to \infty} \frac{1}{N} \cdot S_{\text{min}}(\Phi \otimes N) \]

3. Non-additivity of \( \bar{S}_{\text{min}}(\cdot) \) implies non-additivity of \( C(\cdot) \). \(^9\)

\(^9\)See, for example, [Fukuda]
Finding counterexamples

1. Can we find concrete examples for additivity violation?
2. Concrete counterexamples for the following additivity violation were found: ¹⁰

\[ S_{p,\text{min}}(\Phi \otimes \Phi) < S_{p,\text{min}}(\Phi) + S_{p,\text{min}}(\Phi) \quad p > 2 \]

Here,

\[ S_{p,\text{min}}(\Phi) = \min_{\rho} S_p(\Phi(\rho)) \]

where \( S_p \) is the Renyi \( p \)-entropy.

3. Concrete counterexamples for \( 1 \leq p \leq 2 \) are still open.

¹⁰ [Grudka, M. Horodecki, Pankowski]
Important examples for additivity

1. Entanglement-breaking channels: [Shor][King]
2. Depolarizing channels: [King]
3. Unital qubit channels: [King]

For the above channels \( \Phi \), we know that, with any channel \( \Omega \),

\[
S_{\text{min}}(\Phi \otimes \Omega) = S_{\text{min}}(\Phi) + S_{\text{min}}(\Omega)
\]
\[
\chi(\Phi \otimes \Omega) = \chi(\Phi) + \chi(\Omega)
\]

In particular, with any channel \( \Omega \),

\[
\bar{S}_{\text{min}}(\Phi \otimes \Omega) = \bar{S}_{\text{min}}(\Phi) + \bar{S}_{\text{min}}(\Omega)
\]
\[
C(\Phi \otimes \Omega) = C(\Phi) + C(\Omega)
\]