

## Invisibility Cloaking in 2D with Conformal Mapping

Fermat's Principle: „Light travels along optical shortest paths“ (1662)

Geometry of light obeys a different measure for distance than in Euclidean Space

Speed of light is not the same everywhere:

Consider two uniform media and a light ray travelling through both. What is the relation between angles at the boundary surfaces? We know that light is slowed down by different indexes

Applying Fermat's Principle, this derives Snell's Law:

Generalization of Snell's Law for non-uniform media:

**Optical paths of light can be defined in dependence of the Refractive Index Profile (RIP)  $n(x,y)$  of the medium it travels through.**

Idea of Achieving Invisibility:

Create a finitely big device, altering the path lengths of the light traveling through it resulting all rays being bent around smaller area inside the medium. How is this index defined?

Introduction of complex analysis: let  $z$  be a complex variable,  $w$ . A function  $w(z)$  would describe a transformation of ray trajectories in 2D if the RIP is altered.

What properties must  $w$  have?

- $w$  for big  $z$ : as we can only alter the RIP in a finite area
- $w$  should define our transformation of RIP independent of time. Using the Helmholtz Equation – a time independent form of the wave equation – we can deduce find:

So if we choose  $w$  independent of  $z^*$  and  $w$  independent of  $z$ , then the Helmholtz Equation holds true for  $w$  as well with

with

Choosing  $n'=1$ , our RIP-transformation will be given by

- $w$  alters the measure of distance, not angles. Transformations of this type are conformal
- Media that act as conformal maps are called transformation media and in Complex analysis  $w$  is **holomorphic**, characterized by the **Cauchy-Riemann Equations**

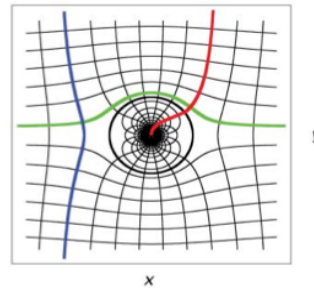
Concept of Riemann sheets: holomorphic functions (all except the Möbius-Transformation) will project onto the whole complex plane more than once creating so-called Riemann sheets. This means that different areas in 2D can be mapped onto different complete 2D sheets.

The first transformation that we will look at is the Zhoukovski Transform:

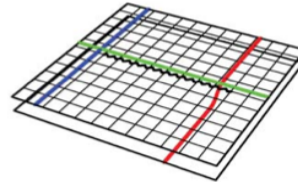
Already we can notice that  $w$  is holomorphic, as it obeys the Cauchy Riemann Equations and therefore conformal with the inverse function:

$w$  maps the complex plane on two Riemann sheets that are connected where the root disappears (2 and -2). Where we draw the trajectory between the sheets is arbitrary but as in the sketch below it is chosen like a straight line between those connection points. This line is equivalent to the unit circle in our  $z$ -plane and hence, everything outside the unit circle will be projected on the first Riemann sheet in the  $w$ -space and everything inside the unit circle on the second Riemann sheet.

z-space: Light traveling in directions like the blue or green one, are bent around the unit circle and continue on their path as if they were not bent. Only the red one enters the unit circle



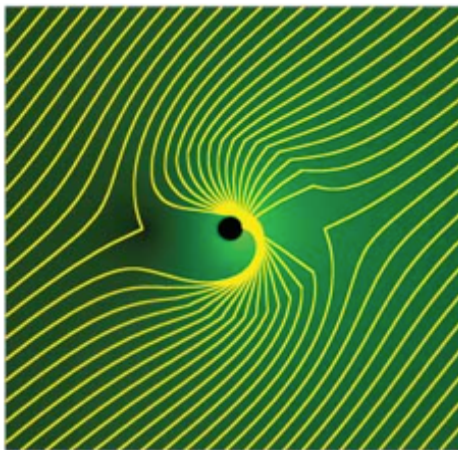
-space: RIP  $n'$  is uniform all lines that don't 'hit the wall' travel as if they travel through air. Those that do (red line) switch sheets at the trajectory (black, ruffled line) and continue on the lower sheet and are lost.



However, we can overcome this problem by creating a different mapping from the interior of the unit circle to the lower Riemann sheet. When the light rays enter the interior of the unit circle, this is analogous to them entering the lower Riemann sheet in the  $-space$ . So what we need to achieve is a RIP on the interior of the unit circle, described by an mapping that forms elliptical paths for all rays entering the lower sheet and redirects them to the point they entered the lower sheet at the same angle.

A possible RIP could be the Kepler Profile:

So with the RIP for the outer part of the unit circle and the Kepler Profile for inside we will get a device that should bend light around a small area inside the unit circle from all directions. The following picture shows a sketch of that device with light rays coming in from a direction similar to the red line figure 3. The ray trajectories are yellow, the brightness of the green background indicates the Refractive Index at that point.



Sources:

Geomtery of Light – Ulf Leonhardt

Wolfram Alpha

Science Magazine

Jason Ross on Riemann Surfaces – Youtube

Prof.Dr. Bornemann – Lecture on Complex Analysis, SS 2012

Wikipedia

Lookout:

Tolga Erin, Nicolas Stenger – 3D Invisibility Cloak at Optical Wavelengths