Winning the improbable

Birthday Paradox

P(Two out of 23 men have birthday on the same date) = $1- \frac{365 \times 364 \times \cdots \times 343}{365^{23}} = 1 - 0.4927... > 0.5$

Two Envelopes

Game: 2 envelopes X, Y with \$x and \$ y=2x
Player: chose one (A), decides: switch/ keep

Naive switching argument:
b = 2a ⇔ P(a smaller) = P(a = x) = ½ b = ½ a ⇔ P(a greater) = P(a = y) = ½
E(B|A=a) = P(a=y)* ½ a + P(a=x) * 2a = ½ * ½ a + ½ * 2a = 5/4 a > a ∀ a => switch in every case
Problem now: P(A|B) = 5/4 b > b => switch again ...

Calculation based on fixed x instead of variable a
P(x=a) = P(x=a/2) = ½
E(B|X=x) = P(x=a)*2x + P(x=a/2)*x = 3/2 x
E(A|X=x) = (x+y) – E(B|X=x) = 3x - 3/2 x = 3/2 x

With prior density:
Probability distributions for every possible value
Possible: E(B|A=a) > a ∀ a
Probability distribution for X: fixed
=> for A,B fixed (since P[(A,B)=(X,Y)]=P[(A,B)=(Y,X)] = ½)

[Example in the appendix]

Infinite Game: E(B|A=a) > a ∀a => E(B) > E(A) or E(B) = E(A) = ∞

Same prior distribution for A and B => same expected value => E(B) = E(A) = ∞
Mathematical guarantee for better chances after switching only: E(A), E(B) finite
Reality: limited amount of money in the world

Bayes (Thomas Bayes) - Degree of belief

Bayesian Theorem
P(Hn|E) * P(E) = P(Hn ∩ E) = P(E|Hn) * P(Hn)
H1,...,Hk: Hypothesis beneath one is true
⇨ P(Hn|E) ∝ P(E|Hn) * P(Hn)
Bayesian statistics
Application of the theorem in a wider interpretation of degree of belief
Criticized (inexact results, too many conclusions out of observations, often failure)

Newcomb

Box A	Box B
\$ 1,000	\$ 1,000,000 or \$ 0 after prediction algorithm

Realist:
Choice after prediction, free will => take AB
Dominance principle: taking AB is always better than only B => use this strategy

Fearful:

Expected-utiliy principle: Prediction algorithm always right: P(predict B, take AB) = P(predict AB, take B) = 0
=> only possibilities:\$ 1,000 and \$ 1,000,000 => take B

Appendix

Example prior density

$p(n) := P(x=\$2^n) = \frac{2^n}{3^{n+1}}\ \forall\ n \in N_0 \quad ; \sum_{n=0}^{\infty} p(n) = 1$

Case 1: a = $1=2^0$ => b = 2a=2 => E(B|a=1) > a

Case 2: a>1, $a=2^m$ ($m \in N$)

=> $(X,Y) = (2^m,2^{m+1})$ or $(X,Y) = (2^{m-1},2^m)$

$P((X,Y) = (2^{m-1},2^m)\,|\,a=2^m)=$

$$\frac{\frac{P((2^{m-1},2^m))}{2}}{\frac{P((2^{m-1},2^m))}{2}+\frac{P((2^m,2^{m+1}))}{2}} = \frac{P((2^{m-1},2^m))}{P((2^{m-1},2^m))+P((2^m,2^{m+1}))} = \frac{p(m-1)}{p(m-1)+p(m)} =$$

$$\frac{\frac{2^{m-1}}{3^m}}{\frac{2^{m-1}}{3^m}+\frac{2^m}{3^{m+1}}} = \frac{2^{m-1}*3}{2^{m-1}*3+2^m} = \frac{3}{2+3} = 3/5$$

$P((X,Y) = (2^m,2^{m+1})\,|\,a=2^m)=1-3/5 = 2/5$

$E(B|A=2^m)=3/5 * 2^{m-1} + 2/5 * 2^{m+1} = 11/10 * 2^m > 2^m = a$

⇨ E(B|A=a) > a $\forall a$ => always switch

Sources

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http://en.wikipedia.org/wiki/Two_envelopes_problem