Mind Reading and Universal Cycles

Does the presenter really possess the power of extrasensory perception?

1. Card Trick

The performer has a deck of 31 cards. Each of 5 audience members removes cards from the case, drops the case on the floor and then gives the deck a straight cut at a random position. The last spectator takes the top card. The deck is then passed back, so each of the 5 spectators take the current top card. The performer now asks, “This may sound strange, but would each of you please look at your card, make a mental picture, and try to send it to me telepathically?“ At this is done the performer concentrates and appears confused: “You‘re doing a great job, but there is too much information coming in for me to make sense of. Would all of you who have a red card please stand up and concentrate?“ Now the performer knows all cards.

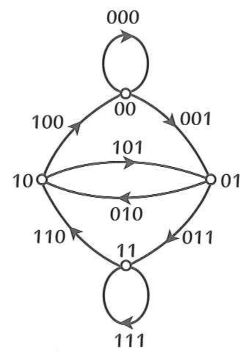
<u>Possible order of the cards:</u> (31 cards, A, ..., 8 of each suit, except of 8C)
- Write each card in binary number with 5 digits
- First two digits represent the suit (C = 00, S = 01, D = 10, H =11)
- last three digits are the value in binary numbers mod 8 (i.e. A = 001, 2 = 010, ... , 7 = 111, 8 = 000)
- e.g. 4H = 11100, 8S = 01000, 00010 = 2C
- the received color pattern represent the first card
- the next card can be computed by taking the last 4 digits of the card before as the first 4 and adding the first and the third digit of the card before mod 2 to receive the 5th digit (e.g. 6S = 01110 -> next card: 11101 = 5H)

2. Universal Cycles

2.1. De Bruijn cycles

A *De Bruijn cycle* with window length *k* is a cyclic zero/one sequence of length 2^k such that every *k* consecutive digits appear just once, e.g. 00111010 is a De Bruijn sequence of length *3*

De Bruijn Graph G_k: Graph with the vertices being the strings of zero/ one symbols of length k-1. An edge is going from one vertex x to another vertex y if there is a zero/one length of length *k* that has *x* at its left and *y* as its right. A Eulerian Cycle in G_k gives a De Bruijn Cycle of window length *k.*



G_3

Number of De Bruijn Cycles of window length *k:* $2^{2^{k-1}-k}$

Construction (Greedy-algorithm)

Given *m* symbols $e_1, \dots, e_m$.

I. Each of the first *k-1* symbols is chosen equal to e_1.

II. The symbol a_m to be added to the sequence (2)

$$a_1\, a_2 \dots a_{k-1}\, a_k \dots a_{m-k+1} \dots a_{m-1}$$
$$a_1 = a_2 = \dots = a_{k-1} = e_1\,,\ m \geq k$$

where the *a*‘s stand for the *e*‘s in a certain order, is the e_i with the greatest subscript consistent with the requirement that the section $a_{m-k+1} \dots a_{m-1}a_m$ duplicate no previously occurring section of *n* symbols in (2).

III. Rule II is first applied for $m = k$ (in which case $a_m = a_k = e_m$) and is then applied repeatedly until a further application is impossible.

Example:
k=3, $m = 2$ with $e_1 = 0, e_2 = 1$
001110100 => De Bruijn Cycle: 00111010

Applications of de Bruijn Cycles
- Robotic vision

- Cryptography
- DNA sequencing

2.2.U-Cycles in general

Let $\mathscr{F}_n$ be a Family of combinatorial objects of 'rank *n*' with their number $l := |\mathscr{F}_n|$. Each $F \in \mathscr{F}_n$ is specified by a sequence $\langle x_1, \ldots, x_n \rangle$, where $x_i \in A$, for some fixed alphabet *A*.

$U = (a_0, a_1, \ldots, a_{m-1})$ is a *universal cycle* for $\mathscr{F}_n$ (or *U-cycle* for short) if $\langle a_{i+1}, \ldots, a_{i+n} \rangle$, $0 \leq i < m$, runs through each element of $\mathscr{F}_n$ exactly once, where index addition is performed modulo *n*.

In the case of De Bruijn cycles the Family is defined as the following:

$$\mathscr{F}_n = B_n = \{0, 1\}^n = \{(x_1, \ldots, x_n) \mid x_i \in \{0, 1\}, 1 \leq i \leq n\}, m=2^n$$

2.3.Other Universal Cycles

2.3.1.Permutations

Let S_n be the set of all *n!* permutations of $\{1, 2, \ldots, n\}$. If $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $\bar{b} = (b_1, b_2, \ldots, b_n)$ are *n*-tuples of distinct integers we will say $\bar{a}$ and $\bar{b}$ are *order-isomorphic* if

$$a_i < a_j \Leftrightarrow b_i < b_j$$

e.g. for n=5*:* (1, 3, 2) and (2, 5, 3) are order-isomorphic.

A U-Cycle $U_n = (a_0, a_1, \ldots, a_{n!-1})$, $a_i \in \{1, 2, .., N\}$ for S_n will be a n!-tuple such that each $\sigma \in S_n$ is order-isomorphic to exactly one block $(a_{i+1}, a_{i+2}, \ldots, a_{i+n})$. (index addition is performed modulo n!)
e.g. the elements of S_3 are 123, 132, 213, 231, 312, 321, so a U-Cycle for S_3 is 145243

2.3.2.Partitions

Let P_n be the set of partitions of the n-element set {1, 2, ... , n}.
A U-Cycle will be a sequence composed of symbols from the set A = {a, b, c, ...} containing every partition exactly once. A block represents a partition, in which *i* and *j* are in the same group if the *i*th and the *j*th symbol are the same (e.g. for *n=5* abacc represents the partitions 13|2|45).
e.g. there are 52 partitions of {1, 2, 3, 4, 5}, a U-Cycle on the alphabet {C, S, D, H, J} is DDDDDCHHHCCDDCCCHCHCSHHSDSSDSSHSDDCHSSCHSHDHSCHSJCDC
The existence of such U-cycles depends on *n.* For example, there is no U-cycle for *n=3*.

2.3.3.*k*-sets of a *n*-set

$\begin{bmatrix} n \\ k \end{bmatrix}$:= family of all *k*-element subsets of an n-element set $\{0, 1, \ldots, n-1\}$.

U-cycle: (k=3, n=8):
02456145712361246703671345034601250135672560234723570147.
Each subsequence abc represents {a, b, c}

<u>Existence</u>: If *k* does not divide $\binom{n-1}{k-1}$, $\begin{bmatrix} n \\ k \end{bmatrix}$ divide, has no U-cycle

3. References

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