Mean-field evolution of fermions with Coulomb interaction

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in collaboration with
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General goal: study the dynamics of the low energy states

$$\begin{cases} i\partial_t \psi_{N,t} = H_N \psi_{N,t} \, , \\ \psi_{N,0} = \psi_N \, . \end{cases}$$

N is large $\Longrightarrow$ look for **scaling regimes** in which the evolution can be well approximated by an **effective dynamics**.

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$$H_N = \sum_{j=1}^{N} \left[-N^{\frac{2}{3}} \Delta_{x_j} - \frac{N^{\frac{4}{3}}}{|x_j|} \right] + N^{\frac{1}{3}} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|}$$

$$= N^{\frac{4}{3}} \left\{ \sum_{j=1}^{N} \left[-N^{-\frac{2}{3}} \Delta_{x_j} - \frac{1}{|x_j|} \right] + \frac{1}{N} \sum_{i<j}^{N} \frac{1}{|x_i - x_j|} \right\}$$

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Ground state $\simeq$ Slater determinant (Bach '92, Graf & Solovej '94)

$$\psi_{\rm Slater}(x_1,\dots,x_N) = \frac{1}{\sqrt{N!}}\det(f_j(x_i))_{i,j\le N}\,, \quad \{f_j\}_{j\le N} \text{ o.n.s. in } L^2(\mathbb{R}^3).$$

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Theorem

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  The **Vlasov equation** is the next degree of approximation.
  The Wigner transform of $\omega_{N,t}$ for $N \to \infty$ solves the Vlasov equation

  $$\partial_t W_t(x,v) + v \cdot \nabla_x W_t(x,v) = \nabla(V * \rho_t)(x) \cdot \nabla_v W_t(x,v)$$

Previous results

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- 2016 Coulomb potential with different scalings:
  - Bach, Breteaux, Petrat, Pickl, Tzaneteas;
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$$\mathrm{tr}\,|\gamma^{(1)}_{N,t} - \omega_{N,t}| \lesssim \langle \mathcal{U}_N(t)\,\Omega\,,\, \mathcal{N}\,\mathcal{U}_N(t)\,\Omega\rangle$$

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- **Control on the fluctuations**

  Bound on the expected number of excitations of the Slater determinant

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Fefferman – de la Llave representation

$$(\star) \quad \frac{1}{N}\int dx\, dy\, \frac{1}{|x-y|} \langle \mathcal{U}_N(t)\Omega,\ a(\omega_{t,x})a(\omega_{t,y})a(1-\omega_{t,y})a(1-\omega_{t,x})\mathcal{U}_N(t)\Omega\rangle$$

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Smooth version of Fefferman–de la Llave representation for the Coulomb potential:

$$\frac{1}{|x-y|} = C\int_0^\infty \frac{dr}{r^5}\int dz\, \chi_{(r,z)}(x)\, \chi_{(r,z)}(y)\,, \quad \chi_{(r,z)}(x) = e^{-|x-z|^2/r^2}$$

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Insert it in $(\star)$:

$$\frac{C}{N}\int dx\, dy \int_0^\infty \frac{dr}{r^5}\int dz\, \chi_{(r,z)}(x)\, \chi_{(r,z)}(y)$$
$$\times \langle \mathcal{U}_N(t)\Omega,\ a(\omega_{t,x})a(\omega_{t,y})a(1-\omega_{t,y})a(1-\omega_{t,x})\mathcal{U}_N(t)\Omega\rangle$$

$$= \frac{C}{N}\int_0^k dr \int dx\, dy\, dz \cdots + \frac{C}{N}\int_k^\infty dr \int dx\, dy\, dz \,\ldots$$

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