Bose Particles in a Box: Convergent Expansion of the Ground State in the Mean Field Limiting Regime

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Motivations and Background

- $N$ Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = L^d$
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H = - \sum_i \Delta_i + \frac{1}{\rho} \sum_{i<j} \phi(x_i - x_j)
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where \( i, j \) run from 1 to \( N = \rho|\Lambda| \)
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- Other regimes: Gross-Pitaevskii
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- Results on the ground state energy (Lieb-Solovej, Erdos-Yau-Schlein, Giuliani-Seiringer)
- Results on the excited spectrum: mean field limit (Seiringer, Lewin-Nam-Sefarty-Solovej), diagonal limit (Derezinski-Napiorkowsky)
- Results on Bose-Einstein condensation (Lieb-Seiringer-Yngvason, Lewin-Nam-Rougerie, Seiringer-Nam-Rougerie, Boccato-Brennecke-Cenatiempo-Schlein)
- Perturbative renormalization group approach: (Benfatto) in space dimension $d=3$, order by order control of the Schwinger functions as $|\Lambda|\to\infty$ and with uv cut-off; recent progress for $d=2$ using Ward identities (Castellani et al., Cenatiempo-Giuliani)
- Rigorous functional integral (Balaban-Feldman-Knoerrer-Trubowiz)
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Summary

1. Definition of the model: Hamiltonian in second quantization, *Particle preserving* Bogoliubov Hamiltonian and three-modes systems

2. A novel application of Feshbach map: Multi-scale analysis in the occupation numbers of particle states

3. Convergent expansion of the ground state of a three-modes Bogoliubov Hamiltonian

4. Outlook
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( \( \Delta \) with periodic boundary conditions)

\[
H := \int (\nabla a^*)(\nabla a)(x) dx + \\
\frac{1}{2\rho} \int \int a^*(x)a^*(y)\phi(x - y)a(y)a(x) dx dy
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Model

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\]

- \( a^*(x), a(x) \) operator-valued distributions on

\[
\mathcal{F} := \Gamma (L^2 (\Lambda, \mathbb{C}; dx)) \quad |\Lambda| = L^d
\]

CCR:

\[
[a^#(x), a^#(y)] = 0, \quad [a(x), a^*(y)] = \delta(x - y)1_\mathcal{F}
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- \(a(x) = \sum_{j \in \mathbb{Z}^d} a_j e^{ik_j \cdot x} \)

\[ k_j := \frac{2\pi}{L} j, \quad j = (j_1, j_2, \ldots, j_d), \quad j_1, j_2, \ldots, j_d \in \mathbb{Z} \]

CCR:

\[ [a^#_j, a^#_{j'}] = 0, \quad [a_j, a^*_j] = \delta_{j,j'} . \]
Assumptions on the two-body potential

- The pair potential $\phi(x - y)$ is a bounded, real-valued function that is periodic, i.e., $\phi(w) = \phi(w + jL)$ for $j \in \mathbb{Z}^d$
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- **Strong interaction potential regime:** The ratio $\epsilon_j := \frac{k^2}{\phi_j}$ is sufficiently small
Model: Particle Preserving Bogoliubov Hamiltonian

\[ H := \int (\nabla a^*)(\nabla a)(x) dx + \]
\[ + \frac{1}{2\rho} \int \int a^*(x)a^*(y)\phi(x - y)a(y)a(x) dx dy \]

is restricted to \( F^N \equiv \) subspace of \( F \) with \( N \) particles (\( N \) even)

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\[ V \equiv \) cubic and quartic terms in the nonzero modes
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\[ H^B_j := (k_j^2 + \frac{\phi_j}{\rho|\Lambda|}a_0^*a_0)(a_j^*a_j + a^*_{-j}a_{-j}) \]
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\[ H^B_j := (k_j^2 + \frac{\phi_j}{N} a_0^*a_0)(a_j^*a_j + a_{-j}^*a_{-j}) + \frac{\phi_j}{N} \left\{ a_0^*a_j a_{-j} + a_j^*a_{-j}^*a_0 a_0 \right\} \]
Model: Particle Preserving Bogoliubov Hamiltonian

\[
H := \int (\nabla a^*)(\nabla a)(x)\,dx +
\]
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+ \frac{1}{2\rho} \int \int a^*(x)a^*(y)\phi(x-y)a(y)a(x)\,dx\,dy
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H = H^B + V + C_N
\]
\[
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\]
\[
H^B := \frac{1}{2} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} H_j^B
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H_j^B := (k_j^2 + \frac{\phi_j}{N} a_0^* a_0)(a_j^* a_j + a_{-j}^* a_{-j})
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Model: Particle Preserving Bogoliubov Hamiltonian

- $H$ is restricted to $\mathcal{F}^N \equiv$ subspace of $\mathcal{F}$ with exactly $N$ particles ($N$ even)
- $H = H^B + V$
  - $V \equiv$ cubic and quartic terms in the nonzero modes
- $H^B := \frac{1}{2} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} H^B_j$
- Three-modes Bogoliubov Hamiltonian

\[
H_j^{(0)}
\]

\[
H_j^B := (k_j^2 + \frac{\phi_j}{N} a_0^* a_0)(a_j^* a_j + a_{-j}^* a_{-j})
+ \left\{ \frac{\phi_j}{N} a_0^* a_j a_{-j} \right\}
+ \left\{ \frac{\phi_j}{N} a_j^* a_{-j}^* a_0 a_0 \right\}

W_j
W_j^*
Why studying the three-modes systems?

- In the mean field limiting regime

\[ \inf(H - C_N) \rightarrow E^B \]

with

\[ E^B := \frac{1}{2} \sum_{j \in \mathbb{Z}^d \backslash \{0\}} E_j^B = -\frac{1}{2} \sum_{j \in \mathbb{Z}^d \backslash \{0\}} \left[ k_j^2 + \phi_j - \sqrt{(k_j^2)^2 + 2\phi_j k_j^2} \right] \]
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- In the limit of infinite particle density each couple of modes interacts with the zero mode only

- The thermodynamic limit is already nontrivial for a three-modes system: a large field problem appears
The Feshbach flow and the construction of the ground state are well defined if \( \varepsilon_{j_*} := \frac{k_{j_*}^2}{\phi_{j_*}} \) is sufficiently small and for some \( \nu > \frac{11}{8} \):

\[
\varepsilon_{j_*}^{\nu} \geq \frac{1}{N} \iff \frac{k_{j_*}^2}{\phi_{j_*}} > \left( \frac{1}{N} \right)^{\frac{8}{11}}
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When is this condition fulfilled?
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- at fixed \( \rho \) only if

\[
\left[ \frac{(2\pi j_*)^2}{L^2 \phi_{j_*}} \right]^\nu \geq \frac{1}{\rho L^d}
\]

\( \Rightarrow \quad d \geq 3 \text{ and } L \text{ large enough} \)
Main results three modes system: Regimes and dimensions

- Existence of the fixed point if

\[ \rho \geq \rho_0 \left( \frac{L}{L_0} \right)^{3-d} \]

with \( \rho_0 \) sufficiently large \((L_0 \equiv 1)\)

- If \( d \geq 3 \Rightarrow L < \infty \) can be taken arbitrarily large at fixed (and large) \( \rho \)
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  with \( \rho_0 \) sufficiently large (\( L_0 \equiv 1 \))
  
  - If \( d \geq 3 \Rightarrow L < \infty \), can be taken arbitrarily large at fixed (and large) \( \rho \)

- In the mean field limiting regime, \( z_* \to E_{j_*}^{Bog} \) as \( N \to \infty \)
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  - If \( d \geq 3 \Rightarrow L < \infty \) can be taken arbitrarily large at fixed (and large) \( \rho \)
  - In the mean field limiting regime, \( z_* \rightarrow E_{j*}^{\text{Bog}} \) as \( N \rightarrow \infty \)

- For \( d = 3 \) and \( \rho \geq \rho_0 \), \( z_* \rightarrow -\phi_{j*} \) as \( L \rightarrow \infty \)
If $\psi_{gs}$ ground state of $H$, $\langle \sum_{j \in \mathbb{Z}^d \setminus \{0\}} a_j^* a_j \rangle_{\psi_{gs}}$ stays bounded as $N \to \infty$

$\Rightarrow$ Conjecture: An effective Hamiltonian in a neighborhood of $E_{gs}$ is a multiple of the projection

$$|\eta\rangle \langle \eta|, \quad \eta := \frac{1}{\sqrt{N!}} a_0^* \cdots a_0^* \Omega$$
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Feshbach map ($\mathcal{P} = \mathcal{P}^2$, $\overline{\mathcal{P}} = \overline{\mathcal{P}}^2$, $\mathcal{P} + \overline{\mathcal{P}} = 1_H$)

\[ \mathcal{F}(K - z) := \mathcal{P}(K - z)\mathcal{P} - \mathcal{P}K\overline{\mathcal{P}} \frac{1}{\overline{\mathcal{P}}(K - z)\mathcal{P}} \overline{\mathcal{P}}K\mathcal{P} \]
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Isospectrality: 1) $\mathcal{F}(K - z)$ is bounded invertible on $\mathcal{P}\mathcal{H}$ if and only if $z$ is in the resolvent set of $K$ (on $\mathcal{H}$); 2) $z$ is an eigenvalue of $K$ if and only if $0$ is an eigenvalue of $\mathcal{F}(K - z)$
Selection rules of $H$ w.r.t. $\sum_{j=\pm j^*} a_j^* a_j$

\[ \Rightarrow \text{choose } \mathcal{P}, \overline{\mathcal{P}} \text{ associated with eigenspaces of } \sum_{j=\pm j^*} a_j^* a_j \]
Selection rules of $H$ w.r.t. $\sum_{j=\pm j^*} a_j^* a_j$

$\Rightarrow$ choose $\mathcal{P}$, $\overline{\mathcal{P}}$ associated with eigenspaces of $\sum_{j=\pm j^*} a_j^* a_j$

The Rayleigh-Schrödinger expansion of $\psi_{gs}$ is not under control for strong interaction potentials (thermodynamic limit). Can $\overline{\mathcal{P}}$ help to avoiding small denominator problems?
Pick a couple of interacting modes ($-j^*; j^*$)
Three-modes system

- Pick a couple of interacting modes \((-j^*, j^*)\)

- Study the Hamiltonian \(\hat{H}^B \equiv H^B_{j^*}\)
Three-modes system

- Pick a couple of interacting modes \((-j^*; j^*)\)

- Study the Hamiltonian \(\hat{H}^B \equiv H^B_j\)

- For the purpose of this talk the Hilbert space \(\mathcal{F}^N\) contains only the degrees of freedom \((0; -j^*; j^*)\)
Feshbach Projections for $\hat{H}^B$

- $Q^{(i,i+1)} :=$ the projection (in $\mathcal{F}^N$) onto the subspace spanned by the vectors with $N - i$ or $N - i - 1$ particles in the modes $j^*$ and $-j^*$ → the operator $a_{j^*}^* a_{j^*} + a_{-j^*}^* a_{-j^*}$ has eigenvalues $N - i$ and $N - i - 1$ when restricted to $Q^{(i,i+1)} \mathcal{F}^N$
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- $F^N = Q^{(0,1)} F^N \oplus Q^{(2,3)} F^N \oplus \ldots \oplus Q^{(N-2,N-1)} F^N \oplus \{ \mathbb{C} \eta \}$
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- $\mathcal{F}^N = Q^{(0,1)} \mathcal{F}^N \oplus Q^{(2,3)} \mathcal{F}^N \oplus \ldots \oplus Q^{(N-2,N-1)} \mathcal{F}^N \oplus \{ \mathbb{C} \eta \}$

- $Q^{(>1)} :=$ the projection onto the orthogonal complement of $Q^{(0,1)} \mathcal{F}^N$ in $\mathcal{F}^N$ → $Q^{(>1)} + Q^{(0,1)} = 1_{\mathcal{F}^N}$

- Iteratively, for $i$ even, $2 \leq i \leq N - 2$, define $Q^{(>i+1)}$ the projection such that $Q^{(>i+1)} + Q^{(i,i+1)} = Q^{(>i-1)}$

  $$Q^{(>N-1)} \equiv |\eta\rangle \langle \eta|$$
Flow of Feshbach Hamiltonians for $\hat{H}^B$

Define $\mathcal{P}(i) := Q(>i+1)$, $\overline{\mathcal{P}}(i) := Q(i,i+1)$.
Flow of Feshbach Hamiltonians for $\hat{H}^B$

- Define $\mathcal{P}(i) := Q(\{i+1\})$, $\overline{\mathcal{P}}(i) := Q(i, i+1)$

- Starting from $K_{-2}^B(z) := \hat{H}^B - z$

  \[
  K_i^B(z) := \mathcal{P}(i) K_{i-2}^B(z) \mathcal{P}(i) - \mathcal{P}(i) K_{i-2}^B(z) \overline{\mathcal{P}}(i) \frac{1}{\mathcal{P}(i) K_{i-2}^B(z) \mathcal{P}(i)} \overline{\mathcal{P}}(i) K_{i-2}^B(z) \mathcal{P}(i)
  \]
For $i$ (even)

$$K_i^B(z) := Q^{(>i+1)}(\hat{H}^B - z)Q^{(>i+1)}$$

$$- Q^{(>i+1)} W R_{i,i}(z) \sum_{l_i=0}^{\infty} \left[ \Gamma_{i,i}(z) R_{i,i}(z) \right]^{l_i} W^* Q^{(>i+1)}$$

$$\Gamma_{i+2,i+2}(z) :=$$

$$= Q^{(i+2,i+3)} W R_{i,i}(z) \sum_{l_i=0}^{\infty} \left[ \Gamma_{i,i}(z) R_{i,i}(z) \right]^{l_i} W^* Q^{(i+2,i+3)}$$

$$\Gamma_{2,2}(z) := Q^{(2,3)} W R_{0,0}(z) W^* Q^{(2,3)}$$
Range of the spectral parameter $z$

- Spectrum of $\hat{H}^B \equiv H_{j*}^B$ as $N \to \infty$ (Seiringer):
  - the ground state energy tends to
    \[
    E_{j*}^B := - \left[ k_{j*}^2 + \phi_{j*} - \sqrt{(k_{j*}^2)^2 + 2\phi_{j*} k_{j*}^2} \right]
    \]
    \[
    E_{j*}^B \to -\phi_{j*} \quad \text{as} \quad \epsilon_{j*} \to 0
    \]
  - the first excited eigenvalue tends to
    \[
    E_{j*}^B + \sqrt{(k_{j*}^2)^2 + 2\phi_{j*} k_{j*}^2}
    \]
Range of the spectral parameter $z$

- Spectrum of $\hat{H}^B \equiv H^B_j$ as $N \to \infty$ (Seiringer):
  - the ground state energy tends to
    
    $$E^B_{j^*} := -\left[ k_{j^*}^2 + \phi_{j^*} - \sqrt{(k_{j^*}^2)^2 + 2\phi_{j^*} k_{j^*}^2} \right]$$

    $$E^B_{j^*} \to -\phi_{j^*} \quad \text{as} \quad \epsilon_{j^*} \to 0$$

  - the first excited eigenvalue tends to
    
    $$E^B_{j^*} + \sqrt{(k_{j^*}^2)^2 + 2\phi_{j^*} k_{j^*}^2}$$

- Question:
  Can we control the flow for $z < E^B_{j^*} + \sqrt{(k_{j^*}^2)^2 + 2\phi_{j^*} k_{j^*}^2}$?
General Term

For $i$ (even)

\[ K_i^B(z) := Q^{(>i+1)}(\hat{H}^B - z)Q^{(>i+1)} \]

\[ - Q^{(>i+1)} W R_{i,i}^B(z) \sum_{l_i=0}^{\infty} \left[ \Gamma_{i,i}(z) R_{i,i}^B(z) \right]^{l_i} W^* Q^{(>i+1)} \]

\[ (R_{i+2,i+2}^B(z))^{1/2} \Gamma_{i+2,i+2}^B(z)(R_{i+2,i+2}^B(z))^{1/2} := \]

\[ = (R_{i+2,i+2}^B(z))^{1/2} W R_{i,i}^B(z) \sum_{l_i=0}^{\infty} \left[ \Gamma_{i,i}(z) R_{i,i}^B(z) \right]^{l_i} W^*(R_{i+2,i+2}^B(z))^{1/2} \]

\[ \Gamma_{2,2}^B(z) := Q^{(2,3)} W R_{0,0}^B(z) W^* Q^{(2,3)} \]
Key estimates to control the Feshbach flow

\[ z \leq E^B_{j^*} + \phi^* \sqrt{\epsilon^2_{j^*} + 2\epsilon_{j^*}} , \]

where

\[ a\epsilon_{j^*} := O(\epsilon_{j^*}) , \]
\[ b\epsilon_{j^*} := O(\sqrt{\epsilon_{j^*}}) , \]
\[ c\epsilon_{j^*} := O(\epsilon_{j^*}) . \]
Key estimates to control the Feshbach flow

\[ z \leq E_{j*}^B + \phi_{j*} \sqrt{\epsilon_{j*}^2 + 2\epsilon_{j*}}, \]

\[ R_{i,i}^B(z) = \frac{Q(i,i+1)}{Q(i,i+1)(\hat{H}_B - z)Q(i,i+1)} \frac{1}{Q(i,i+1)} \]
Key estimates to control the Feshbach flow

\[ z \leq E_{j^*}^B + \phi_{j^*} \sqrt{\epsilon_{j^*}^2 + 2\epsilon_{j^*}}, \]

\[ R_{i,i}^B(z) = Q^{(i,i+1)} \frac{1}{Q^{(i,i+1)}(\hat{H}_B - z)Q^{(i,i+1)}} Q^{(i,i+1)} \]

key estimate

\[ \| R_{i,i}^B(z) \|^{1/2} W \left[ R_{i-2,i-2}^B(z) \right]^{1/2} \| R_{i-2,i-2}^B(z) \|^{1/2} W^* \left[ R_{i,i}^B(z) \right]^{1/2} \leq \frac{1}{4(1 + a\epsilon_{j^*} - \frac{2b\epsilon_{j^*}}{N-i+1} - \frac{1-c\epsilon_{j^*}}{(N-i+1)^2})} \]

where \( a_{\epsilon_{j^*}} := \mathcal{O}(\epsilon_{j^*}) \), \( b_{\epsilon_{j^*}} := \mathcal{O}(\sqrt{\epsilon_{j^*}}) \), \( c_{\epsilon_{j^*}} := \mathcal{O}(\epsilon_{j^*}) \)
Key estimates to control the Feshbach flow

\[ z \leq E_{j*}^B + \phi_{j*} \sqrt{\epsilon_{j*}^2 + 2\epsilon_{j*}}, \]

\[ R_{i,i}^B(z) = Q^{(i,i+1)} \frac{1}{Q^{(i,i+1)}(\hat{H}^B - z)Q^{(i,i+1)}} Q^{(i,i+1)} \]

key estimate

\[ \| R_{i,i}^B(z) \|^2 W \| R_{i-2,i-2}^B(z) \|^2 \| R_{i-2,i-2}^B(z) \|^2 W^* \| R_{i,i}^B(z) \|^2 \leq \frac{1}{4(1 + a_{\epsilon_{j*}} - \frac{2b_{\epsilon_{j*}}}{N-i+1} - \frac{1-c_{\epsilon_{j*}}}{(N-i+1)^2})} \]

where \( a_{\epsilon_{j*}} := O(\epsilon_{j*}) \), \( b_{\epsilon_{j*}} := O(\sqrt{\epsilon_{j*}}) \), \( c_{\epsilon_{j*}} := O(\epsilon_{j*}) \)

\[ \epsilon_{j*} := \frac{k_{j*}^2}{\phi_{j*}} \]

small but, more importantly, \( \epsilon_{j*}^{\nu} \geq \frac{1}{N} \) for some \( \nu > \frac{11}{8} \)

no small parameter, i.e., no flow for \( \epsilon_{j*} \equiv 0 \)
Key estimates to control the Feshbach flow

- Artificial $\phi_{j*}$-dependent Gap

$$R_{i,i}^B(z) = Q(i,i+1) \frac{1}{Q(i,i+1)(\hat{H}^B - z)Q(i,i+1)} Q(i,i+1)$$
Key estimates to control the Feshbach flow

Artificial $\phi_{j*}$-dependent Gap

\[
R_{i,i}^B(z) = \frac{Q(i,i+1)}{Q(i,i+1)(H^{(0)} + W + W^* - z)Q(i,i+1)}
\]
Key estimates to control the Feshbach flow

- Artificial $\phi_{j*}$-dependent Gap

$$R_{i,i}^B(z) = Q^{(i,i+1)} \frac{1}{Q^{(i,i+1)}(H(0) - z)Q^{(i,i+1)}}$$

$H(0) \geq 0$ and $z \simeq -\phi_{j*}$
Final Feshbach Hamiltonian, Fixed Point, and GS Energy

- $Q^{(>N-1)} = P_\eta = |\eta\rangle\langle\eta|$, hence

$$K_{N-2}^B(z) = f(z)P_\eta$$

$$f(z) = -z$$

$$-\langle\eta, W R_{N-2,N-2}^B(z) \sum_{l=0}^{\infty} \left[ \Gamma_{N-2,N-2}^B(z) R_{N-2,N-2}^B(z) \right]^l W^* \eta \rangle$$
Final Feshbach Hamiltonian, Fixed Point, and GS Energy

- $Q^{(N-1)} = P_\eta = |\eta\rangle \langle \eta |$, hence

$$K_{N-2}^B(z) = f(z)P_\eta$$

$$f(z) = -z$$

$$-\langle \eta, W R_{N-2,N-2}^B(z) \sum_{l=0}^{\infty} \left[ \Gamma_{N-2,N-2}^B(z) R_{N-2,N-2}^B(z) \right]^l W^* \eta \rangle$$

- $f(z)$ is decreasing and there is (only) one point $z_*$ in the interval

$$(-\infty, E_{j_*}^B + \sqrt{\epsilon_{j_*} \sqrt{(k_{j_*}^2)^2 + 2\phi_{j_*} k_{j_*}^2}})$$

such that $f(z_*) = 0$
Final Feshbach Hamiltonian, Fixed Point, and GS Energy

- \( Q^{(\eta > N-1)} = P_\eta = |\eta\rangle\langle\eta| \), hence

\[
K^B_{N-2}(z) = f(z)P_\eta
\]

\[
f(z) = -z
\]

\[
-\langle\eta, W R^B_{N-2,N-2}(z) \sum_{l=0}^{\infty} \left[ \Gamma^B_{N-2,N-2}(z) R^B_{N-2,N-2}(z) \right]^l W^* \eta \rangle
\]

- \( f(z) \) is decreasing and there is (only) one point \( z_* \) in the interval

\[
(-\infty, E^B_{j_*} + \sqrt{\epsilon_{j_*}} \sqrt{(k_{j_*}^2)^2 + 2\phi_{j_*} k_{j_*}^2})
\]

such that \( f(z_*) = 0 \)

- \( f(z_*) = 0 \implies z_* \) is the ground state energy of \( \hat{H}^B \)
Feshbach theory: If $\varphi$ eigenvector of $\mathcal{H}(K - z^*)$ with eigenvalue 0

$$\left[\mathcal{P} - \frac{1}{\mathcal{P}(K - z^*)\mathcal{P}}\overline{\mathcal{P}K\mathcal{P}}\right]\varphi$$

is eigenvector of $K$ with eigenvalue $z^*$.
Convergent expansion (up to any desired precision)

\[ \psi^B_N = \eta - \sum_{j=2}^{N/2} \prod_{r=2j}^{4} \left[ - \frac{1}{Q(N-r,N-r+1) K^B_{N-r-2}(z_*) Q(N-r,N-r+1)} \right] W^* \eta \]

where \( W^*_{N-r,N-r+2} := Q(N-r,N-r+1) W^* Q(N-r+2,N-r+3) \)
Fix $\epsilon_{j^*}$. Then, $\forall \xi > 0 \exists j_{\xi}$, $N_{\xi}$ such that $\forall N > N_{\xi}$ with the property

$$
\psi_N^B = \eta - \frac{1}{Q(N-2,N-1)K_{N-4}^B(z_*)Q(N-2,N-1)}Q(N-2,N-1)W^*\eta
$$

$$
- \sum_{j=2}^{4} \prod_{r=2j}^{4} \left[ - \frac{1}{Q(N-r,N-r+1)K_{N-r-2}^B(z_*)Q(N-r,N-r+1)}W_{N-r,N-r+2}^* \right] \times
$$

$$
\frac{1}{Q(N-2,N-1)K_{N-4}^B(z_*)Q(N-2,N-1)}Q(N-2,N-1)W^*\eta + O(\xi)
$$
Complete Hamiltonian: Control of the cubic and quartic terms

- New first step in each Feshbach flow
Complete Hamiltonian: Control of the cubic and quartic terms

- New first step in each Feshbach flow

- Short range property of the interaction Hamiltonian in the particle states occupation numbers
Complete Hamiltonian: Control of the cubic and quartic terms

- New first step in each Feshbach flow
- Short range property of the interaction Hamiltonian in the particle states occupation numbers
- Semigroup property of the Feshbach map
- $N$ Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = 1$

$$H = -\sum_i \Delta_i^{(x)} + g N^2 \sum_{i<j} \phi(N(x_i - x_j))$$

with $N \to +\infty$
$N$ Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = 1$

$$H = - \sum_i \Delta_i^{(x)} + g N^2 \sum_{i<j} \phi(N(x_i - x_j))$$

with $N \to +\infty$

Rescaling: $y = Nx \Rightarrow N$ particles in a box of volume $|\Lambda| = N^3$

$$H = - N^2 \sum_i \Delta_i^{(y)} + g N^2 \sum_{i<j} \phi(y_i - y_j)$$
Outlook / Gross Pitaevskii limit and beyond

- $N$ Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = 1$

$$H = -\sum_i \Delta_i^{(x)} + g N^2 \sum_{i<j} \phi(N(x_i - x_j))$$

with $N \to +\infty$

- Rescaling: $y = N x$ ⇒ $N$ particles in a box of volume $|\Lambda| = N^3$

$$H = -N^2 \sum_i \Delta_i^{(y)} + g N^2 \sum_{i<j} \phi(y_i - y_j)$$

- Three-modes Hamiltonian

$$H_j^B = \sum_{\pm j} \left( N^2 k_j^2 + g \frac{\phi_j}{N} a_0^* a_0 \right) a_j^* a_j + g \frac{\phi_j}{N} \left\{ a_0^* a_0 a_j a_{-j} + a_j^* a_{-j}^* a_0 a_0 \right\}$$

where $k_j^2 \gtrsim N^{-2}$
N Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = 1$

$$H = - \sum_i \Delta_i^{(x)} + g N^2 \sum_{i<j} \phi(N(x_i - x_j))$$

with $N, g \to +\infty$

Rescaling: $y = N x \Rightarrow N$ particles in a box of volume $|\Lambda| = N^3$

$$H = - N^2 \sum_i \Delta_i^{(y)} + g N^2 \sum_{i<j} \phi(y_i - y_j)$$

Three-modes Hamiltonian

$$H^B_j = g \phi_j \left[ \sum_{\pm j} \left( \frac{N^2 k_j^2}{g \phi_j} + \frac{1}{N} a_0^* a_0 a_j^* a_j + \frac{1}{N} \{ a_0^* a_0^* a_j a_{-j} + a_j^* a_{-j}^* a_0 a_0 \} \right) \right]$$

where $k_j^2 \gtrsim N^{-2} \Rightarrow \frac{N^2}{g} \frac{k_j^2}{\phi_j} > N^{-\frac{8}{11}}$ for $g \lesssim N^{\frac{8}{11}}$
THANK YOU
Key estimates to control the Feshbach flow

Control of the sequence

\[ X_{i+2} := 1 - \frac{1}{4(1 + a_{\epsilon j^*} - \frac{b_{\epsilon j^*}}{N-i+1} - \frac{1-c_{\epsilon j^*}}{(N-i+1)^2})} \frac{1}{X_i} \]

\[ X_0 \equiv 1 \text{ and, } 0 \leq i \leq N - 2 \text{ and even} \]