Mathematik 3 für Physik (Analysis 2) Zentralübung 12

Notiztitel 18.07.2011

176. $A:\mathbb{R}\to\mathbb{R}^{d\times d}$, $c:\mathbb{R}\to\mathbb{R}^d$ stetig

$$\mathcal{T}e^{\int_s^t du\,A(u)} = \mathbb{1} + \sum_{n=1}^{\infty} \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{n-1}} ds_n\, A(s_1)\cdots A(s_n)$$

(a) $$\int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{n-1}} ds_n = \int_s^t ds_1 \cdots \int_s^{s_{n-2}} ds_{n-1}(s_{n-1}-s) = \int_s^t ds_1 \cdots \int_s^{s_{n-3}} ds_{n-2}\,\frac{(s_{n-2}-s)^2}{2}$$

$$= \cdots \int_s^t ds_1\,\frac{(s_1-s)^{n-1}}{(n-1)!} = \frac{(t-s)^n}{n!}.$$

(b) $\mathcal{T}e^{\int_s^t du A(u)}$ ist absolut konvergent. Bew: $\max\limits_{[s,t]} \|A(u)\| =: a < \infty$

$$\left\| \int_s^{t'} ds_1 \cdots \int_s^{s_{n-1}} ds_n\, A(s_1)\cdots A(s_n) \right\| \le \int_s^{t'} ds_1 \cdots \int_s^{s_{n-1}} ds_n\, \|A(s_1)\|\cdots\|A(s_n)\|$$

$$\le a^n \frac{(t'-s)^n}{n!} \le a^n \frac{(t-s)^n}{n!} \quad \text{falls } s\le t'\le t$$

$\Rightarrow$ absolute Konvergenz glm auf $[s,t]$

(c) $A(u) = B\ \forall u\in[s,t]$: $\mathcal{T}e^{\int_s^t du A(u)} = e^{B(t-s)}$

$$\int_s^t ds_1 \cdots \int_s^{s_{n-1}} ds_n\, A(s_1)\cdots A(s_n) = \int_s^t ds_1 \cdots \int_s^{s_{n-1}} ds_{n-1}\, B^n = \frac{(t-s)^n}{n!} B^n$$

$$\Rightarrow \mathcal{T}e^{\int_s^t du A(u)} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(t-s)^n}{n!} B^n = e^{B(t-s)}$$

(d) Beh $$\frac{d}{dt}\mathcal{T}e^{\int_s^t du A(u)} = A(t)\,\mathcal{T}e^{\int_s^t du A(u)}$$

Bew: $$\frac{d}{dt}\mathcal{T}e^{\int_s^t du A(u)} \overset{(*)}{=} \sum_{n=1}^{\infty} \frac{d}{dt} \int_s^t ds_1 \cdots \int_s^{s_{n-1}} ds_n\, A(s_1)\cdots A(s_n)$$

$$= A(t) + \sum_{n=2}^{\infty} \int_s^t ds_2 \cdots \int_s^{s_{n-1}} ds_n\, A(t)A(s_2)\cdots A(s_n)$$

$$= A(t)\left[\mathbb{1} + \sum_{n=1}^{\infty} \int_s^t ds_1 \cdots \int_s^{s_{n-1}} ds_n\, A(s_1)\cdots A(s_n)\right] = A(t)\,e^{\int_s^t du A(u)}$$

(e) Setze $E(s,t) := \mathcal{T} e^{\int_s^t du\, A(u)}$ $\quad + \quad \frac{d}{dt} F(t,t) = \partial_1 F(t,t) + \partial_2 F(t,t)$

Zeige, dass $x(t) = E(0,t)\, x_0 + \int_0^t ds\, E(s,t)\, c(s)$ Lösung ist

von $\dot{x} = Ax + c$, $x(0) = x_0 \in \mathbb{R}^d$

Anw (d) $\frac{\partial}{\partial t} E(s,t) = A(t)\, E(s,t)$, $E(t,t) = \mathbb{1}$

$$\frac{d}{dt} x(t) = A(t)\, E(0,t)\, x_0 + E(t,t)\, c(t) + \int_0^t ds\, A(t)\, E(s,t)\, c(s)$$

$$= A(t) \left(E(0,t)\, x_0 + \int_0^t E(s,t)\, c(s)\, ds \right) + c(t) = A(t)\, x(t) + c(t).$$

177. AWP $\dot{x} = -e^{-x} \sin t$ $\quad x(t_0) = x_0$

(a) Lösungen? $\dot{x} = -\frac{\sin t}{e^x}$ $\quad \int_{x_0}^{x} e^x\, dx = \int_{t_0}^{t} -\sin t\, dt$

$$e^x = \cos t + C$$

Lsg: $x(t) = \log(\cos t + C)$

Integrationskonstante: $x_0 = x(t_0) = \log(\cos t_0 + C)$

$$C = e^{x_0} - \cos t_0 .$$

$x(t) = \log\left(e^{x_0} + \cos t - \cos t_0\right)$ Lsg des AWP

$\forall t \in \mathbb{R}$ definiert falls $x_0 > \log(1 + \cos t_0)$ $\quad$ insbes für $t_0 = 0$ $x > \ln 2$

(b) Konstante der Bewegung: $V(x,t) = e^x - \cos t$, da

$$\frac{\partial V}{\partial x} = e^x, \quad \frac{\partial V}{\partial t} = \sin t$$

(c) Bild

$\dot{x} = -\frac{f(t)}{g(x)} \Rightarrow$ $G(x) - F(t)$ ist Konstante der Bew

falls $G' = g$ $\quad F' = f$

178. Potenzreihenansatz

lineare Dgl N-ter Ordnung, lin, inhomogen, nicht autonom

$$\sum_{n=0}^{N} a_n(t)\, x^{(n)}(t) = b(t) \quad , \quad x^{(j)}(0) = x_j,\ j=0,\dots,N-1 \quad (*)$$

$$a_n(t) = \sum_{k=0}^{\infty} a_{n,k}\, t^k \ , \quad b(t) = \sum_{k=0}^{\infty} b_k t^k \qquad \text{mit} \quad \text{Kgzradius} > 0$$

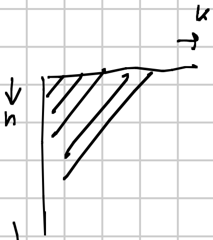
Es gebe eine Lsg von $(*)$ die um 0 in einer Potenzreihe entwickelt werden kann, $x(t) = \sum_{k=0}^{\infty} c_k t^k$. Bestimme c_k

(a) Bestimmungsgl. $c_j = \frac{1}{j!} x_j \ , j = 0,\dots,N-1$

$$c_{N+k} \cdot d_{k,N+k} = b_k - \sum_{m=0}^{N+k-1} c_m d_{k,m} \quad , \quad d_{k,m} = \sum_{l=\max\{m-N,0\}}^{\min(k,m)} a_{m-l,k-l} \frac{m!}{l!}$$

$$x^{(n)}(t) = \sum_{k=n}^{\infty} c_k \underbrace{k(k-1)\cdots(k-n+1)}_{\frac{k!}{(k-n)!}} t^{k-n} = \sum_{k=0}^{\infty} c_{k+n} \frac{(k+n)!}{k!} t^k$$

Cauchy-Produkt: $\sum_{k=0}^{\infty} a_k t^k \sum_{n=0}^{\infty} b_n t^n = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} a_{k-n} b_n \right) t^k$



$$0 = \sum_{n=0}^{N} x^{(n)}(t)\, a_n(t) - b(t) = \sum_{n=0}^{N} \left(\sum_{k=0}^{\infty} c_{k+n} \frac{(k+n)!}{k!} t^k \right) \left(\sum_{l=0}^{\infty} a_{n,l}\, t^l \right) - \sum_{k=0}^{\infty} b_k t^k$$

$$= \sum_{n=0}^{N} \sum_{k=0}^{\infty} \sum_{l=0}^{k} c_{l+n} \frac{(l+n)!}{l!} t^l \ a_{n,k-l}\, t^{k-l} - \sum_{k=0}^{\infty} b_k t^k$$

$$= \sum_{n=0}^{N} \sum_{k=0}^{\infty} \left[\left(\sum_{l=0}^{k} a_{n,k-l}\, c_{l+n} \frac{(l+n)!}{l!} \right) - b_k \right] t^k$$

$l+n \to m'$

$l \to l'$

$$\Rightarrow \forall k \in \mathbb{N}_0: \quad b_k = \sum_{n=0}^{N} \sum_{l=0}^{k} a_{n,k-l}\, c_{l+n} \frac{(l+n)!}{l!} = \sum_{m=0}^{N+k} \sum_{\substack{l=0 \\ 0 \le m-l \le N}}^{k} a_{m-l,k-l}\, c_m \frac{m!}{l!}$$

$$= \sum_{m\geq 0}^{N+h} c_m \underbrace{\sum_{\ell=\max\{m-N,0\}}^{\min\{h,m\}} a_{m-\ell,h-\ell} \frac{m!}{\ell!}}_{=:d_{h,m}}$$

$$\Rightarrow \quad c_{N+h}\, d_{h,N+h} = b_h - \sum_{m=0}^{N+h-1} c_m d_{h,m}$$

$c_0, \dots c_{N-1}$ fest gelegt durch $c_j = \frac{x_j}{j!}$

Falls $d_{h,N+h} \neq 0$ ist c_{N+h} eindeutig fest gelegt

Falls $d_{h,N+h} = 0$ so kann entweder c_{N+h} beliebig gewählt werden oder es gibt keine Lösung.

(b) $\dot{x}(t) = t\, x(t)$, $x(0) = 1$

$$x(t) = \sum_{h\geq 0} c_h t^h \qquad \Rightarrow \quad c_0 = 1$$

$$0 = \dot{x}(t) - t x(t) = \sum_{h\geq 1} h c_h t^{h-1} - \sum_{h\geq 0} c_h t^{h+1}$$

$$= \sum_{h\geq 0} (h+1) c_{h+1} t^{h} - \sum_{h\geq 1} c_{h-1} t^{h}$$

$$= c_1 + \sum_{h\geq 1} \left[(h+1) c_{h+1} - c_{h-1}\right] t^h$$

$$\Rightarrow \quad c_1 = 0, \qquad c_{h+1} = \frac{c_{h-1}}{h+1} \quad h \in \mathbb{N}$$

also $c_{2\ell+1} = 0 \quad \ell \in \mathbb{N}_0$

$$c_{2\ell} = \frac{c_{2\ell-2}}{2\ell} = \dots = \frac{1}{(2\ell)(2\ell-2)\cdots 2} = \frac{1}{2^\ell \ell!}$$

$$x(t) = \sum_{\ell\geq 0} c_{2\ell} t^{2\ell} = \sum_{\ell \geq 0} \frac{1}{\ell!} \left(\frac{t^2}{2}\right)^\ell = e^{\frac{t^2}{2}}$$, Probe $\frac{d}{dt} e^{\frac{t^2}{2}} = t e^{\frac{t^2}{2}}$