Mathematik 3 für Physik (Analysis 2) Zentralübung 8

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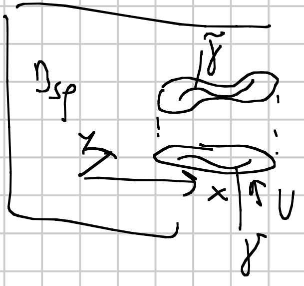
154. Fläche im $\mathbb{R}^3$ parametrisiert durch $\phi: U \to \mathbb{R}^3$,

$U \subset \mathbb{R}^2$ offen. $g: U \to \mathbb{R}^{2\times 2}$ definiert durch

$$\underset{2\times2}{g} := \underset{2\times3}{D\phi^T}\, \underset{3\times2}{D\phi}, \quad D\phi(x,y) \text{ hat vollen Rang}$$



(a) $\gamma: [a,b] \to U$ glatte Kurve. $\tilde{\gamma} := \phi \circ \gamma : [a,b] \to \phi(U)$

Beh: $L(\tilde{\gamma}) = \int_a^b < \underset{2\times1}{\dot{\gamma}(t)}, \underset{2\times2}{g(\gamma(t))}\, \underset{2\times1}{\dot{\gamma}(t)} >^{1/2} dt$.

Bew: $L(\tilde{\gamma}) = \int_a^b \|\dot{\tilde{\gamma}}(t)\| dt = \int_a^b < \underset{3\times2}{D\phi(\gamma(t))}\, \underset{2\times1}{\dot{\gamma}(t)}, \underset{3\times2}{D\phi(\gamma(t))}\, \underset{2\times1}{\dot{\gamma}(t)} >^{1/2} dt =$

$$\overset{(*)}{=} \int_a^b < \underset{2\times1}{\dot{\gamma}(t)}, \underbrace{D\phi(\gamma(t))^T D\phi(\gamma(t))}_{2\times2}\, \underset{2\times1}{\dot{\gamma}(t)} >^{1/2} dt = \int_a^b < \dot{\gamma}(t), g(\gamma(t))\, \dot{\gamma}(t) > dt$$

(*) $x,y \in \mathbb{R}^2$, $A \in \mathbb{R}^{3\times2}$ $\quad < Ax, Ay > = (Ax)^T(Ay) = x^T A^T (Ay) = x^T(A^T A y) =$
$< x, A^T A y >$

(b) G_f Graph von $f: \mathbb{R}^2 \to \mathbb{R}$ Parametrisierung:

$\phi(x,y) = (x, y, f(x,y))$. Berechne $g(x,y)$:

$$D\phi(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1 f & \partial_2 f \end{pmatrix}(x,y). \quad g(x,y) = D\phi(x,y)^T D\phi(x,y) = \begin{pmatrix} 1+(\partial_1 f)^2 & \partial_1 f\, \partial_2 f \\ \partial_2 f\, \partial_1 f & 1+(\partial_2 f)^2 \end{pmatrix}(x,y)$$

(c) Sei $f(x,y) = h\left(\sqrt{x^2+y^2}\right)$ Parametrisierung von G_f:

$$\phi(r,\varphi) = (r\cos\varphi, r\sin\varphi, h(r)) \qquad D\phi(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \\ h'(r) & 0 \end{pmatrix}$$

Metrischer Tensor

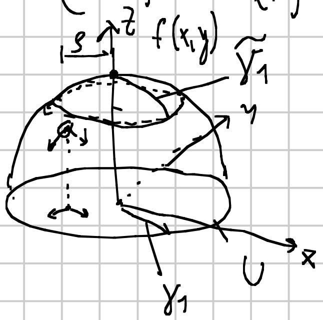
$$g(r,\varphi) = D\phi^T D\phi = \begin{pmatrix} 1+h'(r)^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

$A \in \mathbb{R}^{m\times n}$

$(A^TA)^T = A^T(A^T)^T = A^TA$

(d) Sei nun $h(r) = \sqrt{R^2-r^2}$, $r \in \,]0,R[$

$$h'(r) = -\frac{r}{\sqrt{R^2-r^2}}$$



$$g(r,\varphi) = \begin{pmatrix} \frac{R^2}{R^2-r^2} & 0 \\ 0 & r^2 \end{pmatrix}$$

Kreis $\{x^2+y^2=\varrho^2;\ x^2+y^2+z^2=R^2\}$ $\quad 0<\varrho<R$

$\gamma_1(t) = (t,0) \in \mathbb{R}^2$, $t \in [0,\varrho]$ $\qquad \dot{\gamma}_1(t) = \begin{pmatrix}1\\0\end{pmatrix}$

$$L(\tilde{\gamma}_1) = \int_0^\varrho \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \underbrace{\begin{pmatrix} \frac{R^2}{R^2-t^2} & 0 \\ 0 & t^2 \end{pmatrix}}_{=g(t,\varphi)} \begin{pmatrix}1\\0\end{pmatrix} \right\rangle^{1/2} dt = \int_0^\varrho \frac{R}{\sqrt{R^2-t^2}}\, dt = \overbrace{R \arcsin \frac{\varrho}{R}}^{=:r_i} > \varrho$$

$\gamma_2(t) = (\varrho, t)$, $t \in [0,2\pi]$

$$U = L(\tilde{\gamma}_2) = \int_0^{2\pi} \left\langle \begin{pmatrix}0\\1\end{pmatrix}, g(\varrho,t) \begin{pmatrix}0\\1\end{pmatrix} \right\rangle^{1/2} dt = \int_0^{2\pi} \varrho\, dt = 2\pi\varrho$$

Der Umfang U ist kleiner als $2\pi r_i$

155. $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ linear, $b \in \mathbb{R}^m$

$f(x,y) = b \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m$

Auflösbar nach y? $\Leftrightarrow$ $y = g(x)$, wobei $g: \mathbb{R}^n \to \mathbb{R}^m$

d.i. $f(x, g(x)) = b$

Sei $A \in \mathbb{R}^{m\times(n+m)}$ die darstellende Matrix von f

$$\underset{m\times 1}{f(x,y)} = \underset{m\times(n+m)}{A} \underset{(n+m)\times 1}{\begin{pmatrix} x \\ y \end{pmatrix}} \overset{!}{=} b \quad (*) \quad ; \quad A = \begin{pmatrix} \underbrace{\begin{matrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{matrix}}_{B} & \underbrace{\begin{matrix} A_{1(n+1)} & \cdots & A_{1(n+m)} \\ & & \vdots \\ A_{m(n+1)} & \cdots & \end{matrix}}_{C} \end{pmatrix}$$

Sei $A = (B \; C)$ mit $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{m\times m}$

$(*)$ bedeutet $\underset{m\times n}{B}\underset{n\times 1}{x} + \underset{m\times m}{C}\underset{m\times 1}{y} = \underset{m\times 1}{b} \Leftrightarrow Cy = b - Bx$

$$\overset{C \text{ invertierbar}}{\Leftrightarrow} \quad y = C^{-1}(b - Bx) =: g(x)$$

$$f(x, g(x)) = Bx + C\left(C^{-1}(b - Bx)\right) = b \quad \forall x \in \mathbb{R}^n$$

Gleichung $f(x,y) = b$ ist nach y auflösbar, g.d.w $C \in \mathbb{R}^{m\times m}$ invertierbar

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156, Thermodynamische Beziehungen

$f: C^1(\mathbb{R}^n, \mathbb{R})$. sei $\bar{x} \in \mathbb{R}^n$ eine Lösung von $f(x) = 0$.

Sei diese Gleichung für $j = 1, \dots, n$ nach x_j auflösbar, d.h.

$$x_j = \tilde{X}_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n) \text{ und } \underset{=0}{f(x_1, \dots, x_{j-1}, \tilde{x}_j(\cdots), x_{j+1}, \dots, x_n)}$$

Sei $\partial_j f(\bar{x}) \neq 0$

Beh: $$\frac{\partial x_1}{\partial x_2}\frac{\partial x_2}{\partial x_3} \cdots \frac{\partial x_n}{\partial x_1} = (-1)^n$$

"$\frac{\partial x_1}{\partial x_2}$" steht für $\frac{\partial \tilde{x}_1}{\partial x_2}(\bar{x}_2, \dots, \bar{x}_n)$ $\left(= \partial_1 \tilde{x}_1\right)$

$$\frac{\partial}{\partial x_{j+1}}\left((x_1, \dots x_{j-1}, x_{j+1}, \dots, x_n) \longmapsto \underbrace{f(x_1, \dots, x_{j-1}, \tilde{x}_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), x_{j+1}, \dots, x_n)}_{=0}\right)$$

$$= \left(\frac{\partial f}{\partial x_j} \cdot \frac{\partial \tilde{x}_j}{\partial x_{j+1}} + \frac{\partial f}{\partial x_{j+1}} \right) \left(x_1, \dots, \hat{x}_j, \dots, x_n \right) = 0$$

$$\frac{\partial \tilde{x}_j}{\partial x_{j+1}} = - \frac{\frac{\partial f}{\partial x_{j+1}}(\vec{x})}{\frac{\partial f}{\partial x_j}(\vec{x})} = - \left(\frac{\partial f}{\partial x_j}(\vec{x}) \right)^{-1} \frac{\partial f}{\partial x_{j+1}}(\vec{x})$$

$$\prod_{j=1}^{n} \frac{\partial \tilde{x}_j}{\partial x_{j+1}} \left(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n \right) = \prod_{j=1}^{n} (-1) \cdot \left(\partial_j f(\vec{x}) \right)^{-1} \partial_{j+1} f(\vec{x}) = (-1)^n$$

$$n+1 \equiv 1$$

Bem: (a) Thermodynamik: Zustandsgleichung $f(P,V,T) \overset{\text{ideales Gas}}{=} PV - RT = 0$

P Druck, V Volumen, T Temperatur, R Gas-Konstante

$$\tilde{P}(V,T) = \frac{RT}{V}, \quad \tilde{V}(P,T) = \frac{RT}{P}, \quad \tilde{T}(P,V) = \frac{PV}{R}$$

$$\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$$

$$\frac{\partial \tilde{P}}{\partial V}(V,T) \cdot \frac{\partial \tilde{V}}{\partial T}(P,T) \cdot \frac{\partial \tilde{T}}{\partial P}(V,P) = -1$$

gilt auch für kompliziertere Zustandsgleichungen, z.B. van der Waals

$$P(V - V_0) - R(T + \alpha T^3) = 0$$

$B := (b_1 \cdots b_n) \in \mathbb{R}^{n \times n}$ $\quad b_1, \dots, b_n \in \mathbb{R}^n$ paarweise orthogonal

$$\Rightarrow \quad B^{-1} = \begin{pmatrix} \|b_1\|^{-2} \, b_1^T \\ \vdots \\ \|b_n\|^{-2} \, b_n^T \end{pmatrix}$$