Mathematik 3 für Physik (Analysis 2) Zentralübung 6

Notiztitel 06.06.2011

Blatt 7 145, $f,g \in C^2(\mathbb{R}^3)$ $\left(f,g: \mathbb{R}^3 \to \mathbb{R},\ 2\times \text{ stetig diffbar}\right)$

$v,w \in C^2(\mathbb{R}^3,\mathbb{R}^3)$ $\left(v,w: \mathbb{R}^3 \to \mathbb{R}^3,\ 2\times \text{ stetig diffbar}\right)$

(a) $\nabla(fg) = g\nabla f + f\nabla g$

$\nabla f = \operatorname{grad} f$

$\nabla \cdot v = \operatorname{div} v$

$\nabla \times v = \operatorname{rot} v$

Bew: $(\nabla(fg))_j = \partial_j(fg) = \partial_j f \cdot g + f \cdot \partial_j g = (g\nabla f)_j + (f\nabla g)_j$

(b) $\nabla \cdot (fv) = (\nabla f)\cdot v + f(\nabla \cdot v)$

Bew: $\nabla \cdot (fv) = \sum_j \partial_j (fv)_j = \sum_j \partial_j (f v_j) = \sum_j (\partial_j f) v_j + f \partial_j v_j = \nabla f \cdot v + f \operatorname{div} v$

(c) $\nabla \cdot (v \times w) = w \cdot (\nabla \times v) - v \cdot (\nabla \times w)$

ϵ_{ijh}. $1 = \epsilon_{123} = \epsilon_{231} = \epsilon_{312}$, $-1 = \epsilon_{213} = \epsilon_{132} = \epsilon_{321}$, $= 0$ sonst

$a,b \in \mathbb{R}^3$

$(a\times b)_i = \sum_{jh} \epsilon_{ijh} a_j b_h$

$a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$

$\sum_h \epsilon_{ijh}\epsilon_{hlm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$$\nabla \cdot (v\times w) = \sum_j \partial_j \Big(\sum_{hl} \epsilon_{jhl} v_h w_l\Big) =$$

$$\sum_j \sum_{hl} \epsilon_{jhl}\left[(\partial_j v_h) w_l + v_h \partial_j w_l\right] = \sum_{jhl} \underbrace{\epsilon_{jhl}}_{= \epsilon_{ljh}} w_l \partial_j v_h + \sum_{jhl} \epsilon_{jhl} v_h \partial_j w_l$$

$$= \sum_l w_l \sum_{jh} \epsilon_{ljh} \partial_j v_h - \sum_h v_h \sum_{jl} \epsilon_{hjl} \partial_j w_l = w\cdot(\nabla \times v) - v \cdot (\nabla \times w)$$

(d) $\nabla \cdot (\nabla \times v) = 0$ $\quad (\operatorname{div} \operatorname{rot} v = 0)$

Bew: $\nabla \cdot (\nabla \times v) = \sum_j \partial_j \sum_{hl} \epsilon_{jhl} \partial_h v_l = \sum_{jhl} \epsilon_{jhl} \partial_h \partial_j v_l = -\sum_{jhl} \epsilon_{hjl} \partial_h \partial_j v_l$

$-\nabla\cdot(\nabla\times v)$

(e) $\nabla \times (\nabla f) = 0$ ($\text{rot grad } f = 0$)

Bew: $(\nabla \times \nabla f)_j = \sum_{kl} \epsilon_{jkl} \underbrace{\partial_k \partial_l f}_{=\partial_l \partial_k f} \underset{\text{Schwarz}}{\overset{f \in C^2}{=}} -\sum_{kl} \epsilon_{jlk} \partial_l \partial_k f = -(\nabla \times \nabla f)_j$

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146. $U \subset \mathbb{R}^n$ offen. $F: U \to \mathbb{R}^n$ heißt Gradientenfeld auf U, falls ein $f: U \to \mathbb{R}$ ex. mit $F = \text{grad } f$. f heißt Stammfunktion von F ($-f$ heißt Potential zu F).

(a) Sei $F \in C^1(U, \mathbb{R}^n)$ ein Gradientenfeld. Beh: $\partial_j F_i = \partial_i F_j \quad i,j = 1,\dots,n$ (Integrabilitätsbedingung)

Bew: Sei f Stammfkt von F, d.h. $\text{grad } f = F$ und damit ist $f \in C^2(U)$.

Also ist $\partial_j F_i = \partial_j \partial_i f \overset{\text{Schwarz}}{=} \partial_i \partial_j f = \partial_i F_j$

Jacobimatrix

$$DF(x) = \begin{pmatrix} \partial_1 F_1 & \partial_2 F_1 & \cdots & \partial_n F_1 \\ \partial_1 F_2 & & & \\ \vdots & & & \\ \partial_1 F_n & - & - & \partial_n F_n \end{pmatrix}(x) = \begin{pmatrix} \text{grad } F_1(x)^T \\ \vdots \\ \text{grad } F_n(x)^T \end{pmatrix}$$ ist hier symmetrisch

$\boxed{n=3}$: $\text{rot } F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix} = 0$

$F(x) = Ax$

F besitzt Stammfkt $\Rightarrow$ $\text{rot } F = 0$

(b) $F, G: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad F(x,y) = \begin{pmatrix} y \\ y-x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad G(x,y) = \begin{pmatrix} y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$DF(x_0,y_0): \mathbb{R}^2 \to \mathbb{R}^2$ linear

$$\boxed{\begin{aligned} DF(x_0,y_0) &= F \\ J_F(x_0,y_0) &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}}$$

F ist kein Gradientenfeld

Sei g Stammfkt von $G \Rightarrow \partial_1 g(x,y) = y \Rightarrow g(x,y) = x\cdot y + c(y)$

$\partial_2 g(x,y) = x - y \Rightarrow g(x,y) = x\cdot y - \frac{1}{2}y^2 + \tilde{c}(x)$

Sei $g(x,y) = x\cdot y - \frac{1}{2}y^2$. Dann ist $\operatorname{grad} g = G$

(c) $F: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \quad F(x) = -\frac{x}{\|x\|^h} \quad h \in \mathbb{N}$

ist Gradientenfeld.

Angabe eines Potentials: $h = 1 \quad F(x) = -\frac{x}{\|x\|}$. Potential $V(x) = \|x\|$

$$\left(-\operatorname{grad} V(x)\right)_j = -\frac{x_j}{\|x\|} = F_j(x)$$

$k=2$. $V(x) = \ln\|x\|$, $(\partial_j V)(x) = \left(\partial_j\left(x \mapsto \ln\|x\|\right)\right)(x) =$

$$= \frac{1}{\|x\|}\partial_j \|x\| = \frac{1}{\|x\|}\frac{x_j}{\|x\|} = -F_j(x)$$

$h \geq 3$, $V(x) = -\frac{1}{h-2}\frac{1}{\|x\|^{h-2}}$. $\partial_j V(x) = +\frac{h-2}{h-2}\frac{1}{\|x\|^{h-1}}\frac{x_j}{\|x\|} = -F_j(x)$

Newtonsches Gravitationsfeld: $F(x) = -\frac{GmM}{\|x\|^3}x$ ist ein Gradientenfeld mit Potential $V(x) = -\frac{GmM}{\|x\|}$

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147 $\quad (x_1 + \dots + x_n)^k = \sum_{i \in \{1,\dots,n\}^k} x_{i_1}\cdots x_{i_k} = \sum_{\nu \in \mathbb{N}_0^n, |\nu| = k} \frac{k!}{\nu!}x^\nu$

$x \in \mathbb{R}^n$

1. Gl. $\quad \left(\sum_{j=1}^n x_j\right)^k = \left(\sum_{j_1=1}^n x_{j_1}\right)\cdots\left(\sum_{j_k=1}^n x_{j_k}\right) = \sum_{j \in \{1,\dots,n\}^k} x_{j_1}\cdots x_{j_k}$

Multiindizes $\quad \nu \in \mathbb{N}_0^n \quad \nu = (\nu_1, \dots, \nu_n)$

$|\nu| = \nu_1 + \dots + \nu_n$, $\nu! = \nu_1! \cdots \nu_n!$ $(x_1, \dots, x_n)^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$

Induktion nach n: $n=1$ klar, $n=2$ binomische Formel

$n \geq 2$

$n \to n+1$: $$\left(\left(x_1 + \dots + x_n\right) + x_{n+1}\right)^k = \sum_{l=0}^{k} \binom{k}{l} \left(x_1 + \dots + x_n\right)^l x_{n+1}^{k-l} =$$

$$\overset{IV}{=} \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \sum_{\substack{\nu \in \mathbb{N}_0^n \\ |\nu| = l}} \frac{l!}{\nu!} (x_1, \dots, x_n)^\nu x_{n+1}^{k-l} \underset{\mu = (\nu_1, \dots, \nu_n, k-l)}{=}$$

$$= \sum_{\substack{\mu \in \mathbb{N}_0^{n+1} \\ |\mu| = k}} \frac{k!}{\mu!} \left(x_1, \dots, x_{n+1}\right)^\mu \quad \square$$

$(1+x+y)^4 = 1 + 4x + 4y + 6x^2 + 12xy + 6y^2 + 4x^3 + 12x^2y + \dots$

$\mu =$	400	310	301	220	211	202	130	121	112	103	040	031	022	013	004
$\frac{k!}{\mu!}$	1	4	4	6	12	6	4	12	12	4	1	4	6	4	1