# Mathematatik 3 für Physik (Analysis 2) Zentralübung 5

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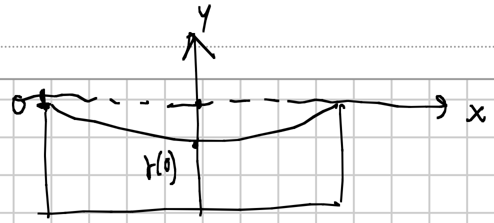



## 13G Kettenlinie

$$\gamma : [-1,1] \to \mathbb{R}^2 \quad \text{mit } \gamma(x) = (x, f(x)),\ f(x) = \frac{1}{a}\left(\cosh(ax) - \cosh a\right)$$

$a > 0$

$$\gamma(0) = \left(0, \frac{1}{a}(1-\cosh a)\right)$$

(a) Länge: $\gamma'(x) = (1, \sinh(ax))$

$$L(\gamma) = \int_{-1}^{1} \sqrt{1+\sinh^2(ax)}\,dx = \int_{-1}^{1} \cosh ax\,dx = \frac{2}{a}\sinh a.$$

(b) Krümmung: $\kappa(x) = \dfrac{f''(x)}{(1+f'(x)^2)^{3/2}}$

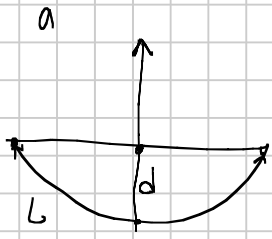
$f'(x) = \sinh(ax)$, $f''(x) = a\cosh(ax) \Rightarrow$

$$\kappa(x) = \frac{a\cosh(ax)}{(1+\sinh^2(ax))^{3/2}} = \frac{a}{\cosh^2 ax}$$

im Scheitel $\kappa(0) = a$

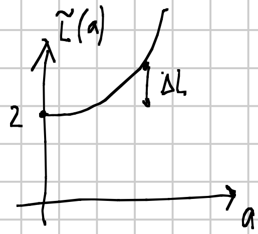
an den Rändern: $\kappa(\pm 1) = \dfrac{a}{\cosh^2 a}$

(c) Durchhang in Abhängigkeit von der Länge.




Durchhang in Abh. von a: $d = \tilde{d}(a) = \frac{1}{a}(\cosh a - 1)$

Bogenlänge: $L = \tilde{L}(a) = \dfrac{2}{a}\sinh a$, $\Delta L = L - 2$

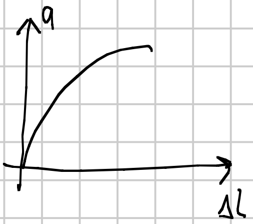



$$\tilde{\tilde{d}}(\Delta L) = \tilde{d}(\tilde{a}(\Delta L)) \qquad \tilde{a} = h^{-1}$$

$$\Delta L = h(a) = \frac{2}{a}\sinh a - 2$$

$$= \frac{2}{a}\left(a + \frac{a^3}{6} + \frac{a^5}{120} + \dots\right) - 2 = \frac{a^2}{3} + \frac{a^4}{60} + \dots$$

$\Delta L \approx \frac{a^2}{3}$ entspricht $a \approx \sqrt{3\Delta L}$




$\tilde{a}(\Delta L) = \sqrt{3}\,\Delta L^{1/2} + \mathcal{O}(\Delta L^{3/2})$

$\tilde{d}(a) = \frac{1}{a}\left(\frac{a^2}{2} + \frac{a^4}{24} + \dots\right) = \frac{a}{2} + \frac{a^3}{24} + \dots \overset{a=\tilde{a}(\Delta L)}{=} \frac{\sqrt{3}}{2}\sqrt{\Delta L} + \mathcal{O}(\Delta L^{3/2})$

$\Delta L = 1mm \Rightarrow d \approx 0{,}87\, m$

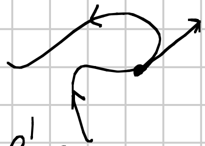
$\Delta L = 10cm \Rightarrow d \approx 8{,}7\, m$

$\Delta L = 1\, cm \Rightarrow d \approx 27{,}4\, m$ (Abw vom exakten Wert ist $\approx 4mm$)

Rückschau: - Konvergenz im $\mathbb{R}^n$, Stetigkeiten.

- Kurvenparametrisierungen $\gamma : [a,b] \to \mathbb{R}^n$ ste diffbar.

  $\gamma', \gamma'', \dots$



Reparametrisierungen $f : [c,d] \to [a,b]$ ste di, $f' > 0$

$\gamma \circ f : [c,d] \to \mathbb{R}^n$ andere Param. der "gleichen" Kurve  $((f^{-1})' > 0)$

Äquivalenzrelation: $\gamma \sim \tilde{\gamma} \Leftrightarrow$ es gibt Param. $f$ mit $f' > 0$ und $\tilde{\gamma} = \gamma \circ f$

Äquivalenzklassen $[\gamma] = \{\tilde{\gamma} \mid \tilde{\gamma} \sim \gamma\}$ "Kurve"

Eigenschaften der Kurve
- Länge, Punkte der Kurve, Orientierung
- Tangeinheitsvektoren, Krümmung, Torsion
- Begl. Zwei-, Dreibein

Parametrisierung nach Bogenlänge gibt es zu jeder Kurve genau eine.

## 140 Differenzierbarkeit

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \frac{xy}{x^2+y^2}$, $(x,y) \neq 0$ und $f(0,0) = 0$

(a) $f$ ist partiell differenzierbar

Bew: Für $(x,y) \neq 0$ ist

$$\partial_1 f(x,y) = \frac{\partial}{\partial x} f(x,y) = f_{,x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= y \frac{y^2 - x^2}{(x^2+y^2)^2}, \quad \partial_2 f(x,y) = x \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$\partial_1 f(0,0) = \left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = 0 = \dots = \partial_2 f(0,0)$$

z.B. $\lim_{y\to 0} \partial_x f(0,y) = \lim_{y \to 0} \partial_1 f(0,y) = \lim \frac{1}{y} = \infty \neq \partial_x f(0,0)$

(b) $f$ ist unstetig: $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \to 0$, aber $f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} \neq f(0,0)$

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## 141.

$(a_{klm})_{k,l,m \in \mathbb{N}_0} \subset \mathbb{R}$ habe nur endl. viele von Null verschiedene Glieder. Sei $f: \mathbb{R}^3 \to \mathbb{R}$ mit

$$f(x) = \sum_{k,l,m=0}^{\infty} a_{klm} x_1^k x_2^l x_3^m = a_{000} + a_{100} x_1 + a_{010} x_2 + a_{001} x_3$$

$$+ a_{200} x_1^2 + a_{020} x_2^2 + a_{002} x_3^2 + a_{110} x_1 x_2 + a_{101} x_1 x_3 + a_{011} x_2 x_3 + \dots$$

(a) Gradient: $\operatorname{grad} f: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad \operatorname{grad} f(x) = \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix}(x)$

$$\partial_1 f(x) = \sum_{k,l,m=0}^{\infty} a_{klm} k x_1^{k-1} x_2^l x_3^m = \sum_{k,l,m=0}^{\infty} \underbrace{\left[a_{(k+1)lm}(k+1)\right]}_{b_{klm}} x_1^k x_2^l x_3^m$$

analog $\partial_2 f(x) = \sum_{klm=0}^{\infty} a_{k(l+1)m}(l+1)\, x_1^k x_2^l x_3^m$, ...

Somit ist
$$\operatorname{grad} f(x) = \sum_{k,l,m=0}^{\infty} \begin{pmatrix} (k+1)\, a_{(k+1)lm} \\ (l+1)\, a_{k(l+1)m} \\ (m+1)\, a_{kl(m+1)} \end{pmatrix} x_1^k x_2^l x_3^m$$

$$\partial_1\partial_1 f(x) = \sum_{klm} (k+1) b_{(k+1)lm}\, x_1^k x_2^l x_3^m = \sum_{klm} (k+1)(k+2)\, a_{(k+2)lm}\, x_1^k x_2^l x_3^m$$

$$\partial_1\partial_2 f(x) = \sum_{klm} (k+1)(l+1)\, a_{(k+1)(l+1)m}\, x_1^k x_2^l x_3^m = \partial_2\partial_1 f(x)$$

analog: $\partial_i\partial_j f(x) = \partial_j\partial_i f(x)$

(b) Beh: $f(x) = f(0) + \langle \operatorname{grad} f(0), x\rangle + \frac{1}{2}\langle x, H_f(0)x\rangle + o(\|x\|^2)$ $\quad$ (sogar $+\mathcal{O}(\|x\|^3)$ )

$$f(0) = a_{000}, \quad \operatorname{grad} f(0) = \begin{pmatrix} a_{100} \\ a_{010} \\ a_{001} \end{pmatrix}, \quad H_f(0) = \begin{pmatrix} \partial_1\partial_1 f & \partial_1\partial_2 f & \partial_1\partial_3 f \\ \partial_2\partial_1 f & \partial_2\partial_2 f & \partial_2\partial_3 f \\ \partial_3\partial_1 f & \partial_3\partial_2 f & \partial_3\partial_3 f \end{pmatrix}(0) =$$
$$= \begin{pmatrix} 2a_{200} & a_{110} & a_{101} \\ a_{110} & 2a_{020} & a_{011} \\ a_{101} & a_{011} & 2a_{002} \end{pmatrix}$$

$$\frac{\left| f(x) - \left( f(0) + \operatorname{grad} f(0)^T x + \frac{1}{2} x^T H_f(0) x \right) \right|}{\|x\|^2} =$$

$$= \frac{1}{\|x\|^2} \left| \sum_{k+l+m\geq 3} a_{klm}\, x_1^k x_2^l x_3^m \right| \leq \frac{1}{\|x\|^2} \sum_{k+l+m\geq 3} a_{klm} \|x\|^{k+l+m} \xrightarrow{\|x\|\to 0} 0$$

$\square$