Introduction to Quantum Information Theory

Jonathan Haas (jonathan.haas@tum.de) 02. July 2015

This handout covers some basic definitions regarding quantum information theory and the HSW theorem. For more detailed information refer to the listed references.

1 Fundamentals

Postulate 1.1. To any quantum-mechanical system there is associated a Hilbert space, called the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.

Definition 1.2 (qubit)**.** A qubit is a vector in a 2-dimensional Hilbert space. We denote the state vector $|\Psi\rangle$ in bra-ket notation

$$|\Psi\rangle = a\,|0\rangle + b\,|1\rangle$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$

Definition 1.3 (density operator)**.** The density operator ρ of a quantum-mechanical system characterized by an ensemble $\{p_i, |\Psi_i\rangle\}_{i=1}^n$ with $\sum_{i=1}^n p_i = 1, p_i \geq 0$ is defined as follows:

$$\rho := \sum_{i=1}^{n} p_i\,|\Psi_i\rangle\,\langle\Psi_i|$$

where $|\Psi_i\rangle\,\langle\Psi_i|$ is the projection onto the one-dimensional space spanned by $|\Psi_i\rangle$.

Definition 1.4 (positive operator-valued measure)**.** A POVM is defined by any partition of the identiy operator into a finite set of positive operators $\{E_a\}$ acting on the Hilbert space $\mathcal{H}$ of the system to be measured and satisfying the following properties:

$$E_a^\dagger = E_a, \ \sum_a E_a = id_{\mathcal{H}}, E_a \geq 0 \ \ \forall a = 1, ..., n$$

A POVM mearurement done on a system which is in a state ρ, produces the a^{th} outcome with propability

$$p(a) = Tr(E_a \rho)$$

2 Quantum channels

Definition 2.1. Let $\mathcal{H}$ and $\mathcal{G}$ be finite dimensional Hilbert spaces. A quantum operation Λ is a linear map that acts in the space of operators of a system and transforms density operators into density operators.

$$\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{G}) \ , \ \rho \mapsto \rho'$$

Properties 2.2. Any quantum operation is a linear completely positive trace-preserving (CPTP) map.

Examples 2.3. Some examples of quantum channels are:

- Bit-flip channel
- Phase-flip channel
- Depolarizing channel

3 Classical information transmission using quantum channels

Let M be a message from a finite set of classical messages $\mathcal{M} = \{1, 2, ..., |\mathcal{M}|\}$, $\rho_M^{(n)}$ the quantum state of the encoded message M using n qubits, Λ the quantum channel that is used n times to transmit the message M, $\delta_M^{(n)} = \Lambda^{\otimes n}(\rho_M^{(n)})$ the received output state, $\{E_M^{(n)}\}_{M\in\mathcal{M}}$ the decoding POVM

Definition 3.1 (rate)**.** The rate of information transmission for this encoding-decoding scheme is the number of bits of classical message that is transmitted per use of the channel. It is given by

$$R := \frac{\log|\mathcal{M}|}{n} \ , \ i.e.\ |\mathcal{M}| \simeq 2^{nR}$$

Definition 3.2. The average probability of error is given by

$$p_{error} = 1 - \frac{1}{2^{nR}} \sum_{M=1}^{2^{nR}} Tr(E_M \delta_M)$$

Definition 3.3. The classical capacity of a quantum channel is the maximum achievable rate, meaning the maximum rate for which exists a sequence of encoding-decoding schemes with

$$p_{error} \xrightarrow{n\to\infty} 0$$

Definition 3.4. A message M is said to be encoded in product states, when $\rho_M^{(n)} = \rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n$ where $\rho_1, \rho_2, ..., \rho_n$ are inputs to the channel on separate uses.

Theorem 3.5 (Holevo-Schumacher-Westmoreland)**.** The product state classical capacity of a quantum channel Λ is given by

$$C^{(1)}(\Lambda) = \mathcal{X}^*(\Lambda)$$

where $\mathcal{X}^*(\Lambda)$ is the Holevo Capacity of the channel and is defined as follows:

$$\mathcal{X}^*(\Lambda) := \max_{\{p_i,\rho_i\}} [S(\Lambda(\sum_i p_i, \rho_i)) - \sum_i p_i S(\Lambda(\rho_i))]$$

and $S(\rho)$ being the **von Neumann Entropy** that is defined as:

$$S(\rho) := -Tr(\rho \ln \rho)$$

References

[1] N. Datta, W. Matthews *Lectures Notes*, Quantum Information Theory, University of Cambridge, 2013.

http://www.qi.damtp.cam.ac.uk/sites/default/files/qit6_13.pdf
http://www.qi.damtp.cam.ac.uk/sites/default/files/qit1314_13.pdf
http://www.qi.damtp.cam.ac.uk/sites/default/files/qit910_13_1.pdf
http://www.qi.damtp.cam.ac.uk/sites/default/files/qit21_13.pdf
http://www.qi.damtp.cam.ac.uk/sites/default/files/qit23_13.pdf