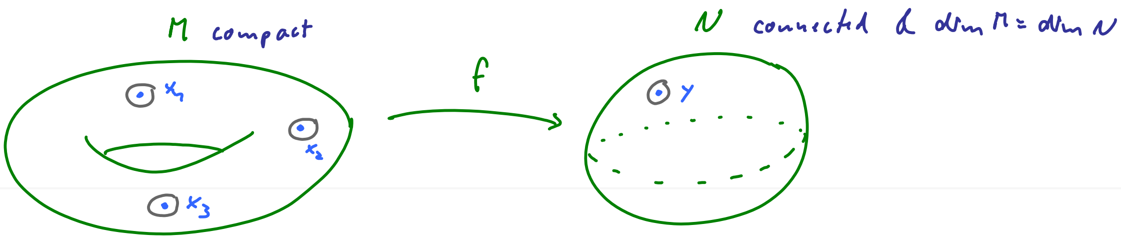


Thm.: Let M, N be smooth manifolds of the same dimension, M compact and N connected (and possibly with boundary). Then for a smooth map $f: M \to N$ with regular value y the <u>"mod 2 degree"</u> of f
$\deg_2(f) := |f^{-1}(y)| \bmod 2$ is independent of the choice of the regular value y and depends only on the smooth homotopy class of f.

proof: Let y & z be two regular values and h an isotopy as in the previous Lemma so that $h_1(y) = z$.

Then z is a regular value of $h_1 \circ f$ and by using homotopy:

$$|f^{-1}(y)| \bmod 2 \underset{y = h_1^{-1}(z)}{=} |(h_1 \circ f)^{-1}(z)| \bmod 2 \underset{h_1 \text{ smoothly homotopic to } h_0 = id}{=} |f^{-1}(z)| \bmod 2$$

If f & g are smoothly homotopic, then $\deg_2(f) = \deg_2(g)$ if there is a common regular value, which always exists by Sard's thm.

$\square$

remarks:
- $\deg_2(f)$ is only defined if $\dim(M) = \dim(N)$, N is connected & M is compact.
- If $\deg_2(f) = 0$ we say that f has <u>"even degree"</u> (and <u>"odd degree"</u> if $\deg_2(f) = 1$).
- beyond mod 2 the notion of a "degree" is only defined if M & N are "oriented".

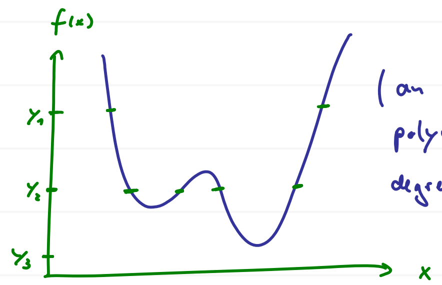
Examples:
- $id: M \to M$ has odd degree (as well as any other bijective map)
- A const. map $f: M \to M$ with $f: x \mapsto c$ has even degree, since any $y \neq c$ is a regular value & $|f^{-1}(y)| = 0$.

- More generally, if $f: M \to N$ is not surjective, f has even degree.

- As a consequence, f has even degree if N is not compact (since $f(M)$ is) or N has boundary (since then f is homotopic to a map that is not surjective - by 'contracting' $f(M)$ & moving it away from the boundary)

- Note that this generalizes the notion of the degree of a polynomial:

$|f^{-1}(y_i)| \bmod 2 = 0$



(an example of a polynomial with even degree)

Winding number mod 2

<u>Def.:</u> Let N be a compact smooth manifold with $\dim N = n$ and let $f: N \to \mathbb{R}^{n+1}$ be a smooth map and $y \in \mathbb{R}^{n+1} \setminus f(N)$.
The "<u>mod 2 - winding number</u>" of f around y is defined as

$$W_2(f,y) := \deg_2(v)$$

where $v: N \to S^n$ is defined as $v(x) := \dfrac{f(x)-y}{\|f(x)-y\|}$.

<u>remarks:</u>

- Note that $|v^{-1}(z)|$ is the number of times $f(x)-y$ points in the same direction as z if we vary x over all of N. So $W_2(f,y)$ is the mod 2 of this number.

- If $N = S^1$ it is easy to see that $W_2(f,y)$ is indeed the mod 2 of the familiar winding number of the closed curve $f(N)$ around y.

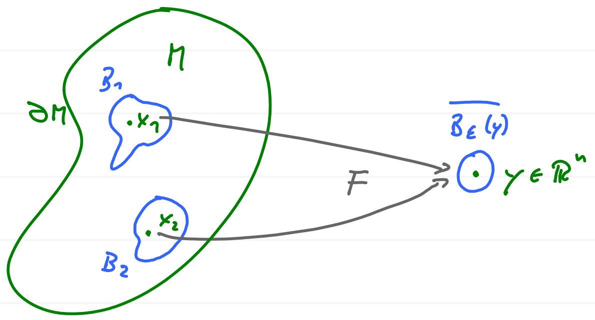
Thm.: Let M be a compact n-dim. smooth manifold with boundary $\partial M \neq \emptyset$, $f: \partial M \to \mathbb{R}^n$ a smooth map and $F: M \to \mathbb{R}^n$ smooth s.t. $F|_{\partial M} = f$.
If $y \notin f(\partial M)$ is a regular value of F, then $F^{-1}(y)$ is a finite set & $W_2(f,y) = |F^{-1}(y)| \bmod 2$.

proof: Suppose first that $y \notin F(M)$.
Then $V: M \to S^{n-1}$, $V(x) := \dfrac{F(x) - y}{\|F(x) - y\|}$ is well defined and $v = V|_{\partial M}$.
By Sard's thm. there exists a common regular value $z \in S^{n-1}$ of v & V.
$V^{-1}(z)$ is then a compact 1-dim. smooth manifold so that

$$\partial(V^{-1}(z)) = \partial M \cap V^{-1}(z) = v^{-1}(z) \text{ contains an even number of points.}$$

So indeed $W_2(f,y) \underset{\substack{\uparrow \\ \text{Def.}}}{=} \deg_2(v) = 0 \underset{\substack{\uparrow \\ y \notin F(M)}}{=} |F^{-1}(y)| \bmod 2$

Now consider the complementary case $F^{-1}(y) \neq \emptyset$
By assumption $y \notin f(\partial M) = F(\partial M)$, so that $\text{Int}\, M \supseteq F^{-1}(y) = \{x_1, \dots, x_k\}$
According to the stack-of-records thm. there are disjoint open neighborhoods $U_j \ni x_j$ and $\tilde{U} \ni y$ s.t. $F|_{U_j}: U_j \to \tilde{U}$ are diffeomorphisms.
Take a closed ball $\overline{B_\varepsilon(y)} \subseteq \tilde{U}$ with radius $\varepsilon > 0$ around y and define $B_j \subseteq U_j$ to be the closed preimages of $\overline{B_\varepsilon(y)}$ under $F|_{U_j}$.



Define $\tilde{M} := M \setminus \left(\bigcup_{j=1}^{k} \operatorname{Int} B_j \right)$, $\tilde{F} := F|_{\tilde{M}}$, $\tilde{V} := V|_{\tilde{M}}$, $\tilde{v} := \tilde{V}|_{\partial \tilde{M}}$.

Then $y \notin \tilde{F}(\tilde{M})$, so we are back at the first case and know that $\tilde{v}$ has even degree.

Moreover, $\tilde{v}^{-1}(z) = v^{-1}(z) \sqcup v_1^{-1}(z) \sqcup \dots \sqcup v_k^{-1}(z)$ where

$$v_j : \partial B_j \to S^{n-1}, \quad v_j(x) := \frac{F(x) - y}{\|F(x) - y\|} = \frac{F(x) - y}{\varepsilon}$$

So $0 = \deg_2(\tilde{v}) = \deg_2(v) + \sum_{j=1}^{k} \deg_2(v_j) \bmod 2$ and therefore

$$\deg_2(v) = \sum_{j=1}^{k} \deg_2(v_j) \bmod 2$$

By the choice of B_j we have that v_j is bijective and therefore $\deg_2(v_j) = 1$.

Hence, $\deg_2(v) = k \bmod 2 = |F^{-1}(y)| \bmod 2$ $\blacksquare$

Suppose $M \subseteq \mathbb{R}^n$ is a compact smooth submanifold and ∂M a connected "hypersurface" (i.e. of co-dimension 1) with inclusion map $f : \partial M \to \mathbb{R}^n$.
Then for $x \notin \partial M$ the value of $w_2(f, x)$ separates $\mathbb{R}^n$ in "inside" ($w_2(f,x) = 1$) & "outside" ($w_2(f,x) = 0$). More generally:

Thm.: (Jordan-Brouwer Separation thm.)

Let X be a compact, connected smooth submanifold of $\mathbb{R}^n$ with co-dim. 1. Then $\mathbb{R}^n \setminus X = A_0 \cup A_1$ where the A_i's are disjoint connected smooth submanifolds of $\mathbb{R}^n$, A_1 is bounded and A_0 and A_1 have point set boundaries $\partial A_0 = \partial A_1 = X$.

proof: → see [Guillemin Pollack] for the idea.

The Borsuk-Ulam theorem

Def.: We call a map $f: S^n \to \mathbb{R}^m$ "odd" if $\forall x \in S^n$: $f(-x) = -f(x)$.

Examples: the antipodal map $x \mapsto -x$, $x \mapsto \sin x$ and polynomials with only odd degree terms

Thm. (Borsuk-Ulam I) Let $f: S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ be an odd smooth map.
Then $W_2(f,0) = 1$.
That is, an odd map must wind around the origin an odd number of times.

here is an alternative formulation:

Thm. (Borsuk-Ulam II) Let $\phi: S^n \to S^n$ be an odd smooth map.
Then $\deg_2 \phi = 1$.

proof that (BU I $\Leftrightarrow$ BU II) $\forall n \in \mathbb{N}$:

BU I $\Rightarrow$ BU II: We may consider $\phi: S^n \to \mathbb{R}^{n+1} \setminus \{0\}$. Then by BU I

$$1 = W_2(\phi, 0) = \deg_2 \left(x \mapsto \frac{\phi(x)}{\|\phi(x)\|} \right) = \deg_2(\phi)$$

BU II $\Rightarrow$ BU I: Setting $\phi(x) = \dfrac{f(x)}{\|f(x)\|}$ we get

$$W_2(f,0) = \deg_2 \left(x \mapsto \frac{f(x)}{\|f(x)\|} \right) = \deg_2(\phi) \qquad \square$$

proof: of BU I by induction. Assume it is true for $n-1$, $n \geq 2$.
Consider S^{n-1} to be the equator of S^n, i.e., embedded by the inclusion map $\iota: (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$. So

$$S^{n-1} \cong \iota(S^{n-1}) = \{ (x,0) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| = 1 \} \subseteq S^n.$$

Define $g : S^{n-1} \to \mathbb{R}^{n+1}$ as $g(x) := f(\iota(x))$, i.e., $g = f|_{\iota(S^{n-1})}$ restricted to the equator. By Sard's thm. $\hat{g} := \frac{g}{\|g\|} : S^{n-1} \to S^n$ and $\hat{f} := \frac{f}{\|f\|} : S^n \to S^n$ have a common regular value, say $y \in S^n$. By symmetry also $-y$ is a common regular value.

Since for $\hat{g}$ the image space has larger dimension than the preimage (and therefore $d\hat{g}_x$ can never be surjective), y being a regular value means that it is not in the image of $\hat{g}$. Consequently, $g(S^{n-1})$ does not intersect the ray $\mathbb{R} \cdot y$.

For $\hat{f}$, on the other hand, y being a regular value implies that

$$|\hat{f}^{-1}(y)| \bmod 2 = \deg_2(\hat{f}) = W_2(f, 0).$$

Using symmetry we get $|\hat{f}^{-1}(y)| = \frac{1}{2} |f^{-1}(\mathbb{R}y)|$. Since f does not map points on the equator into $\mathbb{R}y$, it suffices to restrict to the upper hemisphere $S^n_+ := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \wedge x_{n+1} \geq 0\}$ and to consider $f_+ := f|_{S^n_+}$. Then $\frac{1}{2}|f^{-1}(\mathbb{R}y)| = |f_+^{-1}(\mathbb{R}y)|$ and thus

$$W_2(f, 0) = f_+^{-1}(\mathbb{R}y) \bmod 2 \qquad (1)$$

Now S^n_+ is a manifold with boundary $\partial S^n_+ = \iota(S^{n-1})$ on which we want to apply the induction hypothesis. To this end let $V \subseteq \mathbb{R}^{n+1}$ be the orthogonal complement of $\mathbb{R}y$ and $\pi : \mathbb{R}^{n+1} \to V$ the corresponding orthogonal projection. Then with $h := \pi \circ f_+ : S^n_+ \to V \cong \mathbb{R}^n$ $h|_{\iota(S^{n-1})}$ is odd (since f is odd & π is linear) and 0 is not in its image since $g(S^{n-1}) \cap \{y\mathbb{R}\} = \emptyset$. So by the induction hypothesis we have

$$W_2(h|_{\iota(S^{n-1})}, 0) = 1 \qquad (2)$$

Since $\pm y$ are regular values of $\hat{f}$ it follows (after a little computation starting from $h(x)=0 \Leftrightarrow \hat{f}(x) \in \{\pm y\}$) that 0 is a regular value of h.

We can thus exploit $\underbrace{\text{the main thm. of the previous lecture}}$ & get:

$$W_2\left(h|_{\iota(S^{n-1})}, 0\right) = |h^{-1}(0)| \bmod 2$$

$$\overset{h=\pi\circ f_+}{=} |f_+^{-1}(y\mathbb{R})| \bmod 2$$

$$\overset{(1)}{=} W_2(f, 0)$$

Together with (2) this proves the induction step.

It remains to prove the statement for $n=1$, which can be done by going to the complex plane & will be skipped here. $\square$