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SHEET 5

# Differential Topology: Exercise Sheet 5

**Exercises** (for Dec. 19th and 20th)

### 5.1 Application 1 of Brouwer's fixed point theorem: Perron-Frobenius

In the following, we use the continuous version of Brouwer's fixed-point theorem: every continuous function $f : D^n \to D^n$ has a fixed point. Here $D^n$ is the closed unit ball in $\mathbb{R}^n$. In this exercise, we show the Perron-Frobenius theorem:

> A matrix $A \in \mathbb{R}^{n\times n}$ with $A_{i,j} \geq 0$ for all $i, j \in \{1, \ldots, n\}$ has a non-negative eigenvalue.

In the following, think of $A$ as a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ (expressed in the standard basis).

(a) Show that

$$S^{n-1}_+ = \{x = (x_1, \ldots, x_n) \in S^{n-1} \mid x_j \geq 0 \text{ for all } j = 1, \ldots, n\} \tag{1}$$

is homeomorphic to $D^{n-1}$.

(b) Consider the map $f(x) = Ax/\|Ax\|$ to prove the theorem.

*Solution:*

(a) Note that $S^{n-1}_+ \subset S^{n-1}$. Writing this in terms of polar coordinates, we obtain

$$\begin{aligned}
x_n &= \cos\phi_1 \\
x_{n-1} &= \sin\phi_1 \cos\phi_2 \\
x_{n-2} &= \sin\phi_1 \sin\phi_2 \\
&\vdots \\
x_2 &= \sin\phi_1 \ldots \sin\phi_{n-2} \cos\phi_{n-1} \\
x_1 &= \sin\phi_1 \ldots \sin\phi_{n-2} \sin\phi_{n-1}
\end{aligned}$$

with $\phi_i \in [0, \pi/2]$ for all $i \in \{1, \ldots, n-1\}$ for $x \in S^{n-1}_+$. We can map this to the upper half of $S^{n-1}$ (which we denote by $S^{n-1}_{x_1\geq 0} = \{x \in S^{n-1} | x_1 \geq 0\}$) via the map $\theta : (\phi_1, \ldots, \phi_{n-1}) \mapsto (2\phi_1, \ldots, 2\phi_{n-1})$. Projecting the upper half of $S^{n-1}$ down to $\mathbb{R}^{n-1}$ (by the map $\vartheta : S^{n-1}_{x_1\geq 0} \ni (x_1, x_2, \ldots, x_n) \mapsto (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$) yields the closed unit ball $D^{n-1}$. Composing the maps $\theta$ and $\vartheta$ lets us obtain the desired homeomorphism between $S^{n-1}_+$ and $D^{n-1}$.

(b) Clearly, $f$ maps $S^{n-1}_+$ to $S^{n-1}_+$, thus by part (a), we can identify this with a map $\tilde{f} : D^n \to D^n$. The latter map has to have a fixed point, which corresponds to a $v \in S^{n-1}_+$ such that $f(v) = v$. This means that $Av = ||Av||_2 v$. As $0 \notin S^{n-1}_+$, we have found an eigenvector to the eigenvalue $||Av||_2 > 0$.

**5.2 Counterexample to Brouwer's fixed point theorem in infinite dimensions**

In this exercise, we show that Brouwer's fixed point theorem does not extend to infinite dimensions, i.e. in an infinite-dimensional space, not every continuous function $f : D \to D$ (where $D$ is the closed unit ball in the infinite-dimensional space) has a fixed point.

For example, consider the space $\ell_2$ of square summable sequences, i.e., $x = (x_1, x_2, \ldots) \in \ell_2$ if and only if $x_i \in \mathbb{R}$ and $\|x\|_2^2 = \sum_{i=1}^{\infty} x_i^2 < \infty$ and let $D = \{x \in \ell_2 \mid \|x\|_2^2 \leq 1\}$ denote the closed unit ball in $\ell_2$. Use this space to construct a counter example: find a continuous function $f : D \to D$ that has no fixed point.

*Solution:*

Consider the map $f : D \to \ell_2$ which shifts all elements to the right, i.e. mapping $x = (x_1, x_2, x_3, \ldots) \mapsto (c, x_1, x_2, \ldots)$ for some constant $c \in \mathbb{R}$. It is obviously continuous and has norm $\|f(x)\|_2^2 = c^2 + \|x\|_2^2$ for $x \in D$. Assume that $f$ has a fixed point. Then this fixed point has to be a constant sequence $x = (x_1, x_2, x_3, \ldots)$ where $x_1 = x_2 = x_3 = \ldots = c$. Since the only constant sequence with finite norm is the zero sequence, the fixed point is $x = (0, 0, \ldots)$ which implies $c = 0$. Now choose $c = \sqrt{1 - \|x\|_2^2} \neq 0$ implying $\|f(x)\| = 1$ and hence $f(x) \in D$ for all $x \in D$. Thereby $f : D \to D$ is continuous but has no fixed point.

**5.3 Classification of compact 1-manifolds with smooth embedding**

Let $M$ be a 1-dimensional compact connected manifold with boundary and assume that there is a smooth embedding into some $\mathbb{R}^n$. Show that there must be two boundary points.

*Hint:* Starting from one boundary point, parametrize the manifold in terms of its arc length and show that the arc length is finite.

*Solution:* will follow in January