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Sheet 3

Differential Topology: Exercise Sheet 3

Exercises (for Nov. 21th and 22th)

3.1 Smooth maps

Prove the following:

(a) A map $f : M \to N$ between smooth manifolds $(M, \mathcal{A})$, $(N, \mathcal{B})$ is smooth if and only if for all $x \in M$ there are pairs $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ such that $x \in U$, $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

(b) Compositions of smooth maps between subsets of smooth manifolds are smooth.

Solution:

(a) As the charts of a smooth structure cover the manifold the above property is certainly necessary for smoothness of $f : M \to N$.
To show that it is also sufficient, consider two arbitrary charts $\left(\tilde{U}, \tilde{\phi}\right) \in \mathcal{A}$ and $\left(\tilde{V}, \tilde{\psi}\right) \in \mathcal{B}$. We need to prove that the map $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1} : \tilde{\phi}(\tilde{U}) \to \tilde{\psi}(\tilde{V})$ is smooth. Therefore consider an arbitrary $y \in \tilde{\phi}(\tilde{U})$ such that there is some $x \in M$ such that $x = \tilde{\phi}^{-1}(y)$. By assumption there are charts $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ such that $x \in U$, $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth. Now we choose U_0, V_0 such that $x \in U_0 \subset U \cap \tilde{U}$ and $f(x) \in V_0 \subset V \cap \tilde{V}$ and therefore we can write

$$\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}|_{\tilde{\phi}(U_0)} = \left(\tilde{\psi} \circ \psi^{-1}\right) \circ \left(\psi \circ f \circ \phi^{-1}\right) \circ \left(\phi \circ \tilde{\phi}^{-1}\right) . \qquad (1)$$

As a composition of smooth maps, we showed that $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}|_{\tilde{\phi}(U_0)}$ is smooth. As $x \in M$ was chosen arbitrarily we proved that $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ is a smooth map.

(b) Let $(M, \mathcal{A}), (N, \mathcal{B}), (P, \mathcal{C})$ denote smooth manifolds and $S \subset M$, on which we define smooth functions $f : S \to N$ and $g : f(S) \to P$. We will show that the composition $g \circ f : S \to P$ is smooth as well. For an arbitrary point $x \in S$ there are open neighborhoods $U \subset M$ and $V \subset N$ of $x \in M$ and $y = f(x) \in N$, respectively such that there are smooth extensions $\tilde{f} : U \to N$ and $\tilde{g} : V \to P$. We will prove that $\tilde{g} \circ \tilde{f} : U \cap \tilde{f}^{-1}(V) \to P$ is a smooth map which would give us an extension of $g \circ f$ in a neighborhood of $x \in M$. Then we would be finished as $x \in M$ was chosen arbitrarily.
Consider charts $(U', \phi) \in \mathcal{A}$ and $(W', \psi) \in \mathcal{C}$ with $x \in U'$ and $g \circ f(x) \in W'$ for which we want to prove thath

$$\psi \circ \tilde{g} \circ \tilde{f} \circ \phi^{-1} : \phi\left(U' \cap U \cap \tilde{f}^{-1}(V)\right) \to \psi(V') \cap V \qquad (2)$$

is smooth. To do this take a chart $(V', \nu) \in \mathcal{B}$ restrict it to $V_0 \subset \tilde{f} \circ \phi^{-1}\left(\phi\Big(U' \cap U \cap \tilde{f}^{-1}(V)\Big) \cap V'\right)$ with $f(x) \in V_0$ and insert it as

$$\psi \circ \tilde{g} \circ \tilde{f} \circ \phi^{-1} = \left(\psi \circ \tilde{g} \circ \nu^{-1}\right) \circ \left(\nu \circ \tilde{f} \circ \phi^{-1}\right) . \tag{3}$$

This shows that $\psi \circ \tilde{g} \circ \tilde{f} \circ \phi^{-1} : \phi(V_0) \to \psi \circ \tilde{g} \circ \tilde{f} \circ \phi^{-1}(V_0)$ is smooth as a composition of two smooth maps as $\tilde{f}$ and $\tilde{g}$ are smooth themselves. This finishes the proof.

3.2 Mazur's swindle

The connected sum $\sharp$ is a basic operation on oriented, connected, compact, n-dimensional manifolds. It has a number of interesting properties. One can show that

(a) $M \sharp S^n \simeq M$ (unit element)

(b) $(M \sharp N) \sharp P \simeq M \sharp (N \sharp P)$ (associativity)

(c) $M \sharp N \simeq N \sharp M$ (commutativity)

for n-dimensional manifolds M, N, P where $\simeq$ denotes equal up to homeomorphisms.

Show that the sphere S^n is itself irreducible, i.e. if $S^n \simeq M \sharp N$ for n-dimensional manifolds M, N, then $M, N \simeq S^n$.

Note: You can use the above properties without proof. Note that the associativity also holds for a connected sum of infinitely many topological manifolds.

Solution:
Assume that $S^n \simeq M \sharp N$ for two topological manifolds M, N. We now use associativity of the connected sum of infinitely many topological manifolds:

$$\begin{aligned} S^n &\overset{(a)}{\simeq} S^n \sharp S^n \sharp S^n \sharp \dots \\ &\overset{S^n \simeq M \sharp N}{\simeq} (M \sharp N) \sharp (M \sharp N) \sharp (M \sharp N) \sharp \dots \\ &\overset{(b)}{\simeq} M \sharp (N \sharp M) \sharp (N \sharp M) \sharp \dots \\ &\overset{(c)}{\simeq} M \sharp S^n \sharp S^n \sharp \dots \\ &\overset{(a)}{\simeq} M \end{aligned}$$

By first using commutativity of the connected sum one can prove $N \simeq S^n$ in the same way. This trick is known as Mazur's swindle because of its similarity to the fake proof of $1 = 0$ via Grandi's series

3.3 System of inequalities

Is the set $S := \{x \in \mathbb{R}^3 \mid \sum_{i=1}^3 x_i^3 = 1, \text{ and } \sum_{i=1}^3 x_i = 0\}$ a smooth submanifold of $\mathbb{R}^3$?
Solution:
Take the map $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x_1, x_2, x_3) = \left(\sum_{i=1}^3 x_i^3, \sum_{i=1}^3 x_i\right)$. We have $S = f^{-1}(1, 0)$ and furthermore

$$df_x = \begin{pmatrix} 3x_1^2 & 3x_2^2 & 3x_3^2 \\ 1 & 1 & 1 \end{pmatrix} \tag{4}$$

which is a rank 2 matrix for every $x \in f^{-1}(1, 0)$ (Note that it is rank 1 iff $x_1^2 = x_2^2 = x_3^2$ which is impossible for the values we have). This means that $(1, 0)$ is a regular value and S is a smooth manifold of dimension 1.

3.4 Lie groups

(a) Let G be a Lie group and $H \subset G$ a smooth submanifold that is also a subgroup of G. Show that H is a Lie group as well.

(b) Define the block matrix

$$\sigma := \bigoplus_{k=1}^{n} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

and the **real symplectic group** $\mathsf{Sp}(2n, \mathbb{R}) := \{S \in \mathbb{R}^{2n\times 2n} \mid S\sigma S^T = \sigma\}$. Prove that $\mathsf{Sp}(2n, \mathbb{R})$ with the matrix multiplication and the matrix inversion forms a Lie group. What is the manifold dimension of $\mathsf{Sp}(2n, \mathbb{R})$?

Solution:

(a) As H is a submanifold of G the inclusion map $e : H \to G$ is smooth and an embedding. This has been proven in the lecture as for every point $h \in H \subset G$ there is a chart (U, ϕ) of G around h such that $H \cap U = \phi^{-1}(\mathbb{R}^{n-k})$ for some $k \in \mathbb{N}$. For this chart we have $\phi \circ e \circ \phi|_{H\cap U}^{-1}(x_1, \ldots, x_{n-k}) = (x_1, \ldots, x_{n-k}, 0, \ldots, 0)$ which is smooth. This also shows that e is an embedding. Using the result from the next exercise 3.5(b) we see that $e : N \to e(N)$ is also a diffeomorphism which implies that the maps

$$\mu_H = e^{-1} \times e^{-1} \circ \mu_G \circ e \times e : H \times H \to H \tag{6}$$
$$i_H = e^{-1} \circ i_G \circ e : H \to H \tag{7}$$

are smooth maps as compositions of smooth maps. This finishes the proof.

(b) It follows from $S\sigma S^T \sigma^{-1} = I_{2n}$ that $S^{-1} = \sigma S^T \sigma^{-1}$ which shows that elements in $\mathsf{Sp}(2n, \mathbb{R})$ are invertible, i.e. that $\mathsf{Sp}(2n, \mathbb{R})$ is a subgroup of $\mathsf{GL}(2n, \mathbb{R})$. To show that it is also a smooth submanifold, consider $f : \mathsf{GL}(2n, \mathbb{R}) \to \mathbb{R}^{2n\times 2n}_{\mathsf{skew}} = \{A \in \mathbb{R}^{2n\times 2n} \mid A^T = -A\}$ via $f(S) = S\sigma S^T$ which maps invertible matrices to skew-symmetric matrices. Now we can calculate $df_S(B) = B\sigma S^T + S\sigma B^T = (B\sigma S^T) - (B\sigma S^T)^T$ using $\sigma^T = -\sigma$. Since $B \in \mathsf{GL}(2n, \mathbb{R})$ and σS^T is non singular, we see that f is a surjective map. Therefore σ is a regular value of f which shows that $\mathsf{Sp}(2n, \mathbb{R})$ is a smooth submanifold of $\mathsf{GL}(2n, \mathbb{R})$. The manifold dimension of the group is $\frac{(2n-1)n}{2}$ as this is the dimension of $\mathbb{R}^{2n\times 2n}_{\mathsf{skew}}$. The smoothness of the induced multiplication and inversion maps follows from part (a).

3.5 Immersions and embeddings

(a) Formalize and prove the statement: an immersion is locally an embedding.

(b) Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ denote two smooth manifolds. Show that $f : M \to N$ is an embedding if and only if $f : M \to f(M)$ is a diffeomorphism.

Solution:

(a) Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ denote smooth manifolds of dimensions $\dim M = m$ and $\dim N = n \geq m$ and let $f : M \to N$ be an immersion between the manifolds. The statement can be formalized as follows:

For all $x \in M$ there is a neighborhood U of x such that $f|_U : U \to f(U)$ is an embedding, i.e. an immersion mapping U homeomorphically to $f(U)$.

In order to prove the statement consider a point $x \in M$. We already know that if f is an immersion in every neighborhood. So we only have to find a neighborhood that is mapped homeomorphically to its image by f.

Applying the constant rank theorem to f (as an immersion $\text{rank}(f) = \dim(N)$ at every point $x \in M$) yields two charts $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ such that $x \in U$, $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is given by $\psi \circ f \circ \phi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$. This is the canonical embedding of $\phi(U) \subset \mathbb{R}^m$ into $\mathbb{R}^n$. As ϕ, ψ are homeomorphisms by definition this shows that $f|_U : U \to f(U)$ maps U homeomorphically to U.

(b) If $f : M \to f(M)$ is a diffeomorphism, it is clear that $f : M \to N$ is an embedding. It is trivially a homeomorphism onto $f(M)$ and as a diffeomorphism df_x has to be invertible for all $x \in M$. This shows that it is also an immersion because $\text{rank}(df_x) = \dim(M)$ for all $x \in M$.

If $f : M \to N$ is an embedding, $f : M \to f(M)$ is a smooth homeomorphism by definition. It remains to show that $f^{-1} : f(M) \to M$ is locally smooth. Therefore take an arbitrary $y \in f(M)$ which is mapped to $f^{-1}(y) = x \in M$. Because f is an immersion, we can use the constant rank theorem to obtain charts $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ such that $x \in U$, $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is given by $\psi \circ f \circ \phi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$. Here $m = \dim(M)$ and $n = \dim(N)$. Now we can write $\pi \circ f^{-1} \circ \psi^{-1} : \psi(f(U)) \to \mathbb{R}^m$ as the projection $\pi \circ f^{-1} \circ \psi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_m)$ which is of course smooth. As $(f(U), \psi)$ is a chart of the manifold $f(M)$ in a neighborhood of $y \in f(M)$ and because it is enough to check smoothness or one pair of charts (due to Ex. 3.1(a)) this finishes the proof.