Prof. Dr. M. Wolf
M. Heinze

Differential Topology: Exercise Sheet 6

Exercises (for Jan. 16th and 17th)

6.1 Application 2 of Brouwer's fixed point theorem: existence of a Nash equilibrium

In this exercise you will use Brouwer's fixed point theorem to show the existence of Nash equilibria which appear in game theory.

Formally, an n-player game is given by a tuple (n, f, S). To every player $i = 1, \ldots, n$ we assign a non-empty, finite set of pure strategies S_i. Let $S = S_1 \times \cdots \times S_n$ be the set of all combinations of strategies. Player i may choose a (pure) strategy $x_i \in S_i$, this represents an option that the player may play. The payoff function $f : S \to \mathbb{R}^n$ is defined as $f(x) = (f_1(x), \ldots, f_n(x))$ for $x = (x_1, \ldots, x_n) \in S$. Note that the payoff $f_i(x)$ of player i may not only depend on his or her own choice x_i but also on the strategies of all other players.

We slightly extend this definition to *mixed strategies*: now, every player may choose a probability distribution over pure strategies. Each coordinate in a vector of a mixed strategy M_i is the probability of the corresponding pure strategy. E.g. the vector $(1, 0, \ldots, 0)$ represents playing the first pure strategy in S_i. Then the payoff function $f : M \to \mathbb{R}^n$ is a convex combination of the payoff of pure strategies.

For a strategy $x \in M$ let us introduce the notation $x_{-i} \in M_1 \times \cdots M_{i-1} \times M_{i+1} \times \cdots \times M_n$ to denote the strategies of all players but player i.

A strategy $x^* \in M$ is a *Nash equilibrium* of a game (n, f, M) if no deviation of this strategy by a single player can increase the payoff function of this player, that is for all $i = 1, \ldots, n$ and any $x_i \in M_i$ we find

$$f_i(x^*) \geq f_i(x^*_{-i}, x_i) \ . \tag{1}$$

(a) Define the gain function for player i quantifying how much higher the payoff is of the j-th pure strategy in S_i over a mixed strategy $x_i \in M$ (given the mixed strategies of the other players), that is $g_{i,j} : M \to M$ and

$$g_{i,j}(x) = \max\{0, f_i(e_j, x_{-i}) - f_i(x)\} \ . \tag{2}$$

For $i = 1, \ldots, n$ define $b_i : M \to M_i$ as

$$b_i(x) = \left(\frac{x_{i,1} + g_{i,1}(x)}{1 + \sum_{j=1}^{|S_i|} g_{i,j}(x)}, \ldots, \frac{x_{i,|S_i|} + g_{i,|S_i|}(x)}{1 + \sum_{j=1}^{|S_i|} g_{i,j}(x)} \right) \ . \tag{3}$$

Argue that for $x \in M$ the function $b(x) = (b_1(x), \ldots, b_n(x))$ maps to valid mixed strategies and that it is continuous.

(b) We define the best response of a given strategy as the best option that player i may play given the strategies of the other players, i.e. for $x \in M$ the best response for player i is $r_i(x_{-i}) = \{y \in M_i | f_i(x_{-i}, y) = \max_{z \in M_i} f_i(x_{-i}, z)\}$.

Show that a pure strategy attains maximal payoff, that is for every strategy $x \in M$ and all $i = 1, \ldots, n$ there exists a pure strategy $y_i \in S_i$ in $r_i(x_{-i})$.

(c) Apply Brouwer's fixed point theorem to $b : M \to M$ from (a). Combine this with (b) to show the existence of a Nash equilibrium.

6.2 Embedding of projective spaces

Show that the function $f : \mathbb{R}P^2 \to \mathbb{R}^4$ defined by

$$f\big(q(x_1, x_2, x_3)\big) = (x_1^2 - x_2^2, x_1 x_2, x_1 x_3, x_2 x_3)$$

is an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$. Here, q is the canonical projection onto equivalence classes. We use the definition of $\mathbb{R}P^2$ as $S^2/\sim$ where we identify antipodal points (see exercise 2.1). You may use without proof that q is locally a diffeomorphism (This could be verified using charts).