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Zentrum Mathematik

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Mathematical Basis of Quantum Statistical Physics

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We will first discuss **Exercise 6.4.**, which was left over from the last exercise sheet:
Consider the C^*-algebra $\mathcal{A} := \mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$, where $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, and the two representations $\pi_i : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H}_i)$, $\pi_i(A_1 \oplus A_2) := A_i$ (for $i = 1, 2$). Write down a state $\omega \in \mathcal{A}^*$ which is π_1-normal but not π_2-normal.

7.1. Non-superposable states (proof of Theorem 34)

Let ω_1, ω_2 be *pure* states of a C^*-algebra $\mathcal{A}$. We want to show that ω_1 and ω_2 are *not superposable* if and only if their corresponding cyclic (GNS) representations are *not unitarily equivalent*.

(a) Given a representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ such that ω_1, ω_2 are vector states $\omega_i(A) = \langle \psi_i | \pi(A) \psi_i \rangle$, define Hilbert spaces $\mathcal{H}_i := \overline{\pi(\mathcal{A})\psi_i} \subseteq \mathcal{H}$. Show that $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant subspaces of π and argue that the subrepresentations $\pi_i := \pi|_{\mathcal{H}_i}$ are cyclic (and therefore unitarily equivalent to the GNS representations of the ω_i) and irreducible.

(b) Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear map satisfying $U\pi_1(A) = \pi_2(A)U$ for all $A \in \mathcal{A}$. Using the irreducibility of π_i, show that U^*U and UU^* are both proportional to the identity operator (Hint: Proposition 20). Hence, a nonzero U with $U\pi_1(A) = \pi_2(A)U$ exists if and only if π_1 and π_2 are unitarily equivalent.

(c) Denote by $P_i \in \mathcal{B}(\mathcal{H})$ the orthogonal projections onto $\mathcal{H}_i \subseteq \mathcal{H}$. Show that $P_2\pi(\mathcal{A})'P_1 = \{0\}$ if and only if π_1 and π_2 are not unitarily equivalent.

(d) If π_1 and π_2 are not unitarily equivalent, show that for all $\alpha, \beta \in \mathbb{C}$ the following holds:

$$\langle \alpha\psi_1 + \beta\psi_2 | \pi(A) | \alpha\psi_1 + \beta\psi_2 \rangle \;=\; |\alpha|^2 \langle \psi_1 | \pi(A) | \psi_1 \rangle + |\beta|^2 \langle \psi_2 | \pi(A) | \psi_2 \rangle \qquad \forall A \in \mathcal{A}\,.$$

(e) If π_1 and π_2 are unitarily equivalent, argue that there exists a unitary $V \in \pi(\mathcal{A})'$ such that $P_2VP_1 \neq 0$, and that this implies the existence of $A_1, A_2 \in \mathcal{A}$ such that $\langle \pi(A_2)\psi_2 | V\pi(A_1)\psi_1 \rangle \neq 0$. (Hint: Use that every element in a C^*-algebra can be written as a linear combination of four unitaries, Lemma 2.2.14 in Bratteli/Robinson.)

(f) Defining $\widetilde{\psi}_1 := V\psi_1$, show that $\omega_1(A) = \langle \widetilde{\psi}_1 | \pi(A) \widetilde{\psi}_1 \rangle$. If π_1 and π_2 are unitarily equivalent, show now that the equality in part (d) (with ψ_1 replaced by $\widetilde{\psi}_1$) is violated for $A := A_2^* A_1$ for certain α, β.

7.2. All automorphisms of $\mathcal{B}(\mathcal{H})$ are inner

Let $\mathcal{H}$ be a Hilbert space, and consider the C^*-algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$. Note that any one-dimensional orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ can be written as $P = |\psi\rangle\langle\psi|$ with a unit vector $\psi \in \mathcal{H}$.

(a) Show that an orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ has one-dimensional range if and only if for every $A \in \mathcal{B}(\mathcal{H})$ there exists a constant $c_A \in \mathbb{C}$ such that $PAP = c_A P$.

(b) Fix now a unit vector $\psi \in \mathcal{H}$. Let $\alpha \in \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ be a *-automorphism of $\mathcal{B}(\mathcal{H})$. Show that $\alpha(|\psi\rangle\langle\psi|)$ is a one-dimensional orthogonal projection.

(c) Let $\psi' \in \mathcal{H}$ be any vector such that $\alpha(|\psi\rangle\langle\psi|) = |\psi'\rangle\langle\psi'|$. Define a map $U : \mathcal{H} \to \mathcal{H}$ by $UA\psi := \alpha(A)\psi'$ (for all $A \in \mathcal{A}$). Show that U is well-defined, and $U \in \mathcal{B}(\mathcal{H})$, and that U is unitary.

(d) Show that $\alpha(A) = UAU^*$ for all $A \in \mathcal{B}(\mathcal{H})$.