TECHNISCHE UNIVERSITÄT MÜNCHEN

**Zentrum Mathematik**

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**Mathematical Basis of Quantum Statistical Physics**

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### 6.1. Cyclic and separating sets of vectors

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, and $S \subseteq \mathcal{H}$ a subset of $\mathcal{H}$. Show that the following two statements are equivalent:

(i) $S$ is *cyclic* for $\mathcal{M}$ (meaning that: $\overline{\text{linspan}(\mathcal{M}S)} = \mathcal{H}$).

(ii) $S$ is *separating* for $\mathcal{M}'$ (meaning that: $A \neq B \in \mathcal{M}' \quad \Rightarrow \quad \exists \psi \in S : A\psi \neq B\psi$).

### 6.2. Examples of cyclic and separating vectors

Let $m, n \in \mathbb{N}$, and define $\mathcal{M} := \mathcal{B}(\mathbb{C}^m) \otimes \mathbb{1}_n := \{A \otimes \mathbb{1}_n \,|\, A \in \mathcal{B}(\mathbb{C}^m)\} \subseteq \mathcal{B}(\mathbb{C}^{mn})$, where $\mathbb{1}_n \in \mathcal{B}(\mathbb{C}^n)$ denotes the identity operator on $\mathbb{C}^n$.

(1) Show that $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H} = \mathbb{C}^m \otimes \mathbb{C}^n = \mathbb{C}^{mn}$. What is $\mathcal{M}'$?

(2) Let $\{e_i\}_{i=1}^m$ and $\{f_j\}_{j=1}^n$ be orthonormal bases of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. In the case $m = n$, show that $\psi := \sum_{i=1}^m e_i \otimes f_i \in \mathcal{H}$ is a cyclic and separating vector for $\mathcal{M}$ (i.e., that $S := \{\psi\}$ is a cyclic and separating set for $\mathcal{M}$).

(3) For which values of $m, n$ does $\mathcal{M}$ have a cyclic vector? For which values of $m, n$ does $\mathcal{M}$ have a separating vector?

(4) What happens for $\mathcal{M} := \mathcal{B}(\mathcal{H}) \otimes \mathbb{1}_{\mathcal{K}}$, where $\mathcal{H}, \mathcal{K}$ are any separable Hilbert spaces?

### 6.3. Existence of faithful and normal states

Recall the notions of a faithful state on a $C^*$-algebra, and of a normal state on a von Neumann algebra.

(a) Let $\mathcal{H}$ be a Hilbert space, and consider the von Neumann algebra $\mathcal{B}(\mathcal{H})$. Show that a faithful and normal state on $\mathcal{B}(\mathcal{H})$ exists if and only $\mathcal{H}$ is a separable.

(b) Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Show that the following statements are equivalent:

  (i) $\mathcal{M}$ is *$\sigma$-finite* (meaning that: any family of pairwise orthogonal projections from $\mathcal{M}$ is at most countable).

  (ii) There exists a countable set $S \subseteq \mathcal{H}$ which is separating for $\mathcal{M}$.

  (iii) There exists a faithful normal state on $\mathcal{M}$.

  [Hint for (i)$\Rightarrow$(ii): Consider a maximal family $\{\psi_i\}_{i\in I} \subseteq \mathcal{H}$ such that the subspaces $\mathcal{M}'\psi_i$ are all orthogonal.]

### 6.4. $\pi$-normal states

Consider the $C^*$-algebra $\mathcal{A} := \mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$, where $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, and the two representations $\pi_i : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H}_i)$, $\pi_i(A_1 \oplus A_2) := A_i$ (for $i = 1, 2$). Write down a state $\omega \in \mathcal{A}^*$ which is $\pi_1$-normal but not $\pi_2$-normal.