TECHNISCHE UNIVERSITÄT MÜNCHEN
Zentrum Mathematik

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Mathematical Basis of Quantum Statistical Physics

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Blatt 5
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5.1. Topologies agree on norm-bounded sets (Proposition 29a,b,c)

Show that the weak topology and the σ-weak topology agree on any norm-bounded set $X \subseteq \mathcal{B}(\mathcal{H})$, i.e. on any set $X \subseteq \mathcal{B}(\mathcal{H})$ satisfying $\sup_{x\in X} \|x\| < \infty$.

Remark: The same holds for the relation between the strong and σ-strong topologies, and between the strong* and σ-strong* topologies.

5.2. ~~Strong operator topologies as natural multiplication operator topologies~~

Show that, alternatively to the lecture, the σ-strong topology can also be defined as the locally convex topology arising from the family of seminorms $\{p_T \,|\, T \in \mathcal{LC}(\mathcal{H})\}$, where $p_T : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is defined by $p_T(A) := \|AT\|$.

The above claim (which is the originally posted exercise) seems to be *wrong* (as well as the "Note" that originally appeared here); see the solution sheets for more comments. I apologize for this. I will try to find the correct statement, and post it as a corrected Exercise sheet 5.

5.3. Properties of the commutant

(a) Let $\mathcal{H}$ be a Hilbert space and $S \subseteq \mathcal{B}(\mathcal{H})$ be any subset. Show that the commutant S' is a unital Banach algebra. Show that S' is closed w.r.t. the weak topology (and thus, w.r.t. any of the topologies from Section 2.1 in the lecture). Show that $S \subseteq S''$ and $S' = S'''$. If S is selfadjoint, i.e. if $S = S^* := \{s^* | s \in S\}$, show that S' is a C^*-algebra and even a von Neumann algebra.

(b) Given any subset $S \subseteq \mathcal{B}(\mathcal{H})$, show that $(S \cup S^*)'' := ((S \cup S^*)')'$ is the smallest von Neumann algebra on $\mathcal{H}$ containing S.

5.4. Polar decomposition in von Neumann algebras

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and $A \in \mathcal{M}$.

(a) Show that there exists a positive operator $Q \in \mathcal{B}(\mathcal{H})$, $Q \geq 0$, and an operator $V \in \mathcal{B}(\mathcal{H})$ such that V^*V is the orthogonal projection onto the closed range $\overline{Q\mathcal{H}}$ of Q and such that $A = VQ$.
(When V^*V is an orthogonal projection, V is called a *partial isometry*.)

(b) Show that V and Q are uniquely determined by the above requirements.

(c) Show that $V \in \mathcal{M}$ and $Q \in \mathcal{M}$. (Hint: Consider the unitaries $U \in \mathcal{M}'$, and use that $\{U \in \mathcal{M}' | U \text{ unitary}\}' = \mathcal{M}$.) Note that for C^*-subalgebras $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ which are not von Neumann algebras, V is generally not an element of $\mathcal{M}$.

5.5. Computing the commutant

(i) Compute the commutant $\mathcal{M}'$ of the following set $\mathcal{M}$ of 11×11-matrices of block form:

$$\mathcal{M} \;:=\; \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & \mathcal{O} \end{pmatrix} \,\middle|\, A \in M_2(\mathbb{C}),\ B \in M_3(\mathbb{C}),\ \mathcal{O} \in M_4(\mathbb{C}) \right\} \;\subseteq\; M_{11}(\mathbb{C}) \;,$$

where the last block $\mathcal{O} \in M_4(\mathbb{C})$ denotes the 4×4 zero matrix. (ii) In obvious notation, one may write $\mathcal{M} = (\mathbb{1}_2 \otimes M_2(\mathbb{C})) \oplus (\mathbb{1}_1 \otimes M_3(\mathbb{C})) \oplus \mathcal{O}_4$. How does $\mathcal{M}'$ look in this notation? (iii) Compute $\mathcal{M}'' := (\mathcal{M}')'$. (iv) What is the center of the von Neumann algebra $\mathcal{M}''$?