4.1. GNS construction

For the following C^*-algebras $\mathcal{A}$ and states ω, find the cyclic representation $(\mathcal{H}, \pi, \Omega)$ that the GNS construction yields.

(a) $\mathcal{A} := M_3(\mathbb{C})$, $\omega(A) := \operatorname{tr}\left[\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} A\right]$ for $A \in \mathcal{A}$.

(b) $\mathcal{A} := \mathcal{C}_0([0,1])$, $\omega(f) := 2\int_0^1 x f(x)dx$ for $f \in \mathcal{A}$.

(c) $\mathcal{A} := \mathcal{C}_0([0,2])$, $\omega(f) := 2\int_0^1 x f(x)dx$ for $f \in \mathcal{A}$.

(d) $\mathcal{A} := \mathcal{C}_0([0,2])$, $\omega(f) := f(1)$ for $f \in \mathcal{A}$.

(e) $\mathcal{A} := \mathcal{B}(\mathcal{K})$, $\omega(A) := \langle\psi|A\psi\rangle$ for $A \in \mathcal{A}$, where $\psi \in \mathcal{K}$ with $\|\psi\| = 1$ is fixed in advance.

4.2. Non-vector states

(a) Let $\mathcal{H}$ be a Hilbert space and $\psi \in \mathcal{H}$ with $\|\psi\| = 1$. Then $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$, $A \mapsto \langle\psi|A\psi\rangle$ is called a *vector state* on the C^*-algebra $\mathcal{B}(\mathcal{H})$. Show that ω is a *pure* state (according to the definition given in the lecture, i.e. an extremal point in the set of states $E_{\mathcal{B}(\mathcal{H})}$).

(b) Now let $\mathcal{H} := L^2([0,1], \mu)$ with the Lebesgue measure μ, and consider the operator Q defined by $(Q\psi)(x) := x\psi(x)$ for any $\psi \in \mathcal{H}$ and $x \in [0,1]$ (a.e.). Show that $Q \in \mathcal{B}(\mathcal{H})$, and that $Q = Q^*$, and compute the spectrum $\sigma(Q)$ and the operator norm $\|Q\|$.

(c) According to Theorem 23' (whose proof involves the Hahn-Banach theorem, the Banach-Alaoglu theorem, and the Krein-Milman theorem), there exists a pure state ω on $\mathcal{B}(\mathcal{H})$ with $\omega(Q^*Q) = \|Q\|^2$. Show that such ω cannot be a vector state (see (a)).

4.3. Purity and irreducibility (proof of Theorem 27 in the lecture)

Let $(\mathcal{H}, \pi, \Omega)$ with $\|\Omega\| = 1$ be a cyclic representation of a C^*-algebra $\mathcal{A}$, and define the state $\omega(A) := \langle\Omega|\pi(A)\Omega\rangle$.

(a) Suppose that ω is *not* a pure state. (i) Show that then there exists a positive linear functional $\rho \leq \omega$ with $\rho \neq \lambda\omega\ \forall\lambda \in \mathbb{C}$. (ii) Define an operator T via

$$\langle\pi(B)\Omega|T\pi(A)\Omega\rangle \;=\; \rho(B^*A) \quad \forall A, B \in \mathcal{A}.$$

Why is T well defined on all of $\mathcal{H}$? Why is bounded, i.e. $T \in \mathcal{B}(\mathcal{H})$? (iii) Show that $T \notin \mathbb{C}\mathbb{1}$. (iv) Show that $T \in \pi(\mathcal{A})'$; i.e. show that $T\pi(C) = \pi(C)T\ \forall C \in \mathcal{A}$.

All of this shows: If ω is not pure, then $\pi(\mathcal{A})' \neq \mathbb{C}\mathbb{1}$, which by Proposition 20 means that $(\mathcal{H}, \pi)$ is not irreducible.

(b) Suppose that $(\mathcal{H}, \pi)$ is *not* irreducible, i.e. there exists an orthogonal projection $P \in \pi(\mathcal{A})'$ with $P \notin \{0, \mathbb{1}\}$. Show that $\rho(A) := \langle\Omega|P\pi(A)\Omega\rangle$ defines a positive linear functional with $\rho \leq \omega$, and $\rho \neq \lambda\omega\ \forall\lambda \in \mathbb{C}$. Then, show that this implies that ω is not pure.

This proves the converse of part (a): If $(\mathcal{H}, \pi)$ is not irreducible, then ω is not pure.