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Zentrum Mathematik

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Mathematical Basis of Quantum Statistical Physics

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3.1. Subrepresentations (proof of Proposition 18 from the lecture)

Let $(\mathcal{H}, \pi)$ be a representation of a C^*-algebra $\mathcal{A}$. Recall the definition of *invariant subspace* from the lecture, and show the following:

(1) A closed subspace $\mathcal{K} \subseteq \mathcal{H}$ is invariant if and only if the orthogonal projector $P_\mathcal{K}$ onto that subspace satisfies:

$$\forall A \in \mathcal{A}: \quad \pi(A)P_\mathcal{K} \; = \; P_\mathcal{K}\pi(A) \; .$$

(2) In this case, $(\mathcal{K}, \pi|_\mathcal{K})$ is a representation of $\mathcal{A}$ (called the *subrepresentation*) where $\pi|_\mathcal{K}$ is defined by the restriction to $\mathcal{K} \subseteq \mathcal{H}$, i.e. $(\pi|_\mathcal{K})(A) := \pi(A)|_\mathcal{K}$ for all $A \in \mathcal{A}$.

(3) Furthermore, in this case, the orthogonal complement $\mathcal{K}^\perp \subseteq \mathcal{H}$ of $\mathcal{K}$ is a closed invariant subspace, and we have the direct sum decomposition $(\mathcal{H}, \pi) = (\mathcal{K}, \pi|_\mathcal{K}) \oplus (\mathcal{K}^\perp, \pi|_{\mathcal{K}^\perp})$.

3.2. Cyclic implies nondegenerate

Let $(\mathcal{H}, \pi, \Omega)$ be a cyclic representation of a C^*-algebra $\mathcal{A}$. Show that $(\mathcal{H}, \pi)$ is non-degenerate.

3.3. Properties of the representation of $\mathcal{C}_0(\mathbb{R})$ on the Hilbert space $L^2(\mathbb{R}, \mu)$

Consider the real line $\mathbb{R}$ together with its usual topology and the Lebesgue measure μ on $\mathbb{R}$. Consider the representation $\pi : \mathcal{C}_0(\mathbb{R}) \to \mathcal{B}\left(L^2(\mathbb{R}, \mu)\right)$ of the C^*-algebra $\mathcal{C}_0(\mathbb{R})$, where $\pi(f)$ for $f \in \mathcal{C}_0(\mathbb{R})$ is defined such that for all $\psi \in L^2(\mathbb{R}, \mu)$ the equality $(\pi(f)\psi)(x) := f(x)\psi(x)$ holds for almost all $x \in \mathbb{R}$.

(a) Show that $(L^2(\mathbb{R}, \mu), \pi)$ is a representation, and that it is faithful.

(b) For $f \in \mathcal{C}_0(\mathbb{R})$, compute $\|\pi(f)\|$ directly from its definition. Compare to Proposition 16 from the lecture.

(c) Show that the representation $(L^2(\mathbb{R}, \mu), \pi)$ is *not* irreducible. Give examples of non-trivial closed invariant subspaces and their corresponding orthogonal projections.

(d) Find all orthogonal projections (i.e. elements satisfying $P = P^* = P^2$) in the C^*-algebra $\mathcal{C}_0(\mathbb{R})$.

(e) Show that $(L^2(\mathbb{R}, \mu), \pi)$ is a cyclic representation by giving an example of a cyclic vector. Furthermore, give an example of a non-zero vector that is not cyclic. Compare to Proposition 20 from the lecture.

(f) Using the definition of π directly, show that π is nondegenerate.

3.4. Commutant (we will discuss the problems on the next page before this one)

(i) Compute the commutant $\mathcal{M}'$ of the following set $\mathcal{M}$ of 11×11-matrices of block form:

$$\mathcal{M} \; := \; \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & \mathcal{O} \end{pmatrix} \middle| \; A \in M_2(\mathbb{C}), \; B \in M_3(\mathbb{C}), \; \mathcal{O} \in M_4(\mathbb{C}) \right\} \; \subseteq \; M_{11}(\mathbb{C}) \; ,$$

where the last block $\mathcal{O} \in M_4(\mathbb{C})$ denotes the 4×4 zero matrix. (ii) In obvious notation, one may write $\mathcal{M} = (\mathbb{1}_2 \otimes M_2(\mathbb{C})) \oplus (\mathbb{1}_1 \otimes M_3(\mathbb{C})) \oplus \mathcal{O}_4$. How does $\mathcal{M}'$ look in this notation? (iii) Compute $\mathcal{M}'' := (\mathcal{M}')'$.

3.5. Nondegeneracy, cyclicity, and vector states

Let $(\mathcal{H}, \pi)$ be a representation of a C^*-algebra $\mathcal{A}$

(a) Show that $(\mathcal{H}, \pi)$ is a nondegenerate representation if and only if the linear span of $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$.

(b) Let $(E_\alpha)_\alpha$ be an approximate identity in $\mathcal{A}$. Show: $(\mathcal{H}, \pi)$ is nondegenerate if and only if $\lim_\alpha \pi(E_\alpha)\psi = \psi$ for all $\psi \in \mathcal{H}$ (i.e., in functional analysis terminology, if and only if $\pi(E_\alpha) \to \mathbb{1}_\mathcal{H}$ converges strongly as $\alpha \uparrow$).

(c) If $(\mathcal{H}, \pi)$ is nondegenerate and $\psi \in \mathcal{H}$ with $\|\psi\| = 1$, show that the functional $\omega : \mathcal{A} \to \mathbb{C}$, $\omega(A) := \langle\psi|\pi(A)\psi\rangle$ is a state on $\mathcal{A}$. (Any state of this form is called a *vector state.*)

(d) Let $(\mathcal{H}, \pi, \psi)$ and $(\mathcal{K}, \sigma, \eta)$ be two cyclic representations of $\mathcal{A}$ which correspond to the same state, i.e. which satisfy $\langle\psi|\pi(A)\psi\rangle = \langle\eta|\sigma(A)\eta\rangle$ for all $A \in \mathcal{A}$. Show that both representations are unitarily equivalent.

3.6. Cauchy-Schwarz inequality for states

Let $\omega : \mathcal{A} \to \mathbb{C}$ be a positive linear functional on a C^*-algebra $\mathcal{A}$. Show:

(a) $\forall A, B \in \mathcal{A} : \quad |\omega(A^*B)|^2 \ \leq \ \omega(A^*A)\omega(B^*B)\,.$

(b) $\forall A \in \mathcal{A} : \quad \omega(A^*) \ = \ \overline{\omega(A)}\,.$

3.7. Extending states from non-unital C^*-algebras

Let ω be a positive linear functional on a C^*-algebra $\mathcal{A}$ without identity, and let $\mathcal{A}_1 := \mathbb{C}\mathbb{1} + \mathcal{A}$ be the C^*-algebra obtained by adjoining an identity.

(a) Let $\widetilde{\omega}$ be a functional on $\mathcal{A}_1$ which extends ω, i.e. which satisfies $\widetilde{\omega}|_\mathcal{A} = \omega$. Show that $\widetilde{\omega}$ is positive if and only if there exists $\mu \geq \|\omega\|$ such that $\widetilde{\omega}(A + \lambda\mathbb{1}) := \omega(A) + \lambda\mu$ for all $A \in \mathcal{A}$, $\lambda \in \mathbb{C}$. What is $\|\widetilde{\omega}\|$?

(b) If $\|\omega\| \leq 1$, show that ω has a *unique* extension to a state $\widetilde{\omega}$ on $\mathcal{A}_1$.

3.8. Ideals and approximate identities

For terminology, see Exercise 2.4.

(a) Let $(E_\alpha)_\alpha$ be an approximate identity in a right *-ideal $\mathcal{J}$ of a C^*-algebra $\mathcal{A}$. By Exercise 2.4(b), $\mathcal{J}$ is then also a left ideal. Show that $(E_\alpha)_\alpha$ is an approximate identity of the left ideal $\mathcal{J}$ as well.
(Note in particular: this completes the proof of the existence of a *two-sided approximate identity* in any C^*-algebra.).

(b) Let $\mathcal{J}$ be a closed two-sided ideal. Use the existence of an approximate identity to show that $\mathcal{J}$ is a *-ideal.