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Mathematical Basis of Quantum Statistical Physics

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http://www-m5.ma.tum.de/Allgemeines/MA5109_2013W

2.1. Spectrum in unital Banach algebras

(a) Let $\mathcal{A} := M_2(\mathbb{C})$ be the algebra of complex 2×2-matrices, and $\mathcal{B} := \left\{ \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{A} \,\middle|\, b \in \mathbb{C} \right\}$ be a subalgebra of $\mathcal{A}$. Show that $\mathcal{A}$ and $\mathcal{B}$ are both unital. What are the identity elements $\mathbb{1}_\mathcal{A}$ and $\mathbb{1}_\mathcal{B}$? For $B = \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{B} \subset \mathcal{A}$, compute the spectra $\sigma_\mathcal{B}(B)$ and $\sigma_\mathcal{A}(B)$. Compare to the (corrected) Proposition 8 from the lecture.

(b) Let $\mathbb{D} := \left\{ z \in \mathbb{C} \,\middle|\, |z| \leq 1 \right\}$ be the closed unit disk in $\mathbb{C}$, and define the *disk algebra* $\mathcal{D}$ to be the set of continuous complex-valued functions on $\mathbb{D}$ which are holomorphic in the interior of $\mathbb{D}$, together with pointwise addition and multiplication, and define $\|f\|_\mathcal{D} := \sup \left\{ |f(z)| \,\middle|\, |z| = 1 \right\}$ for $f \in \mathcal{D}$. Argue that $\|\cdot\|_\mathcal{D}$ defines a norm on $\mathcal{D}$ and that it makes $\mathcal{D}$ into a unital Banach algebra. Check that $f \mapsto f^\dagger$, $f^\dagger(z) := \overline{f(\overline{z})}$ $\forall z \in \mathbb{D}$, defines an involution on $\mathcal{D}$, and that this involution is norm-preserving.

(c) By Cauchy's integral formula, each $f \in \mathcal{D}$ is uniquely determined by its values on the unit circle $\partial\mathbb{D}$. Thus, $\mathcal{D}$ can be regarded as a subalgebra of $\mathcal{C} := \mathcal{C}_0(\partial\mathbb{D})$, the Banach algebra of continuous complex-valued functions on $\partial\mathbb{D}$. Consider the function $g \in \mathcal{D} \subseteq \mathcal{C}$ defined by $g(z) := z^2$, and compute the spectra $\sigma_\mathcal{C}(g)$ and $\sigma_\mathcal{D}(g)$! Why does Proposition 8 from the lecture not apply? What does the Spectral Radius Formula (Theorem 4) yield when applied to $\|g\|_\mathcal{D}$ and $\|g\|_\mathcal{C}$?

2.2. Gelfand-Mazur Theorem

Let $\mathcal{A}$ be a unital Banach algebra such that each non-zero element of $\mathcal{A}$ is invertible. Show that, up to renaming of its elements, $\mathcal{A}$ equals $\mathbb{C}$ as a unital algebra. (In more technical terms: Show that there exists a linear and multiplicative and bijective map $\pi : \mathcal{A} \to \mathbb{C}$.)

2.3. Norm inequality

Let $\mathcal{A}$ be a C^*-algebra and $A, B \in \mathcal{A}$ such that AB is normal. Show: $\|AB\| \leq \|BA\|$.

2.4. Ideals

(a) A linear subspace $\mathcal{J}$ of an algebra $\mathcal{A}$ is called a *left ideal* (resp. *right ideal*) if $A \in \mathcal{A}$ and $J \in \mathcal{J}$ implies that $AJ \in \mathcal{J}$ (resp. $JA \in \mathcal{J}$), and a *two-sided ideal* if it is both. Show: (i) Any left (or right) ideal $\mathcal{J} \subseteq \mathcal{A}$ is a subalgebra of $\mathcal{A}$. (ii) When $\mathcal{A}$ has an identity $\mathbb{1}$ and $\mathcal{J}$ is a left (or right) ideal of $\mathcal{A}$, then:

$$\mathcal{J} = \mathcal{A} \quad \Leftrightarrow \quad \mathcal{J} \text{ contains an invertible element} \quad \Leftrightarrow \quad \mathbb{1} \in \mathcal{J}.$$

Thus, *proper ideals* $\mathcal{J} \subsetneq \mathcal{A}$ never contain the identity $\mathbb{1} \in \mathcal{A}$ (but cf. *approximate identities* to be discussed).

(b) If $\mathcal{A}$ is a *-algebra, then a left/right/two-sided ideal $\mathcal{J}$ is called a *left/right/two-sided *-ideal* if $J \in \mathcal{J}$ implies $J^* \in \mathcal{J}$. Show that a left (or right) *-ideal is automatically a two-sided *-ideal. Check that for a two-sided *-ideal $\mathcal{J}$, the *quotient space* $\mathcal{A}/\mathcal{J}$, which consists of the elements $\widehat{A} := A + \mathcal{J}$ for $A \in \mathcal{A}$, becomes a *-algebra when equipped with the operations $\lambda\widehat{A} + \widehat{B} := \widehat{\lambda A + B}$ (for $\lambda \in \mathbb{C}$), $\widehat{AB} := \widehat{A}\widehat{B}$, $(\widehat{A})^* := \widehat{A^*}$.

(c) Let $\mathcal{A}$ be a Banach space, $V \subseteq \mathcal{A}$ a closed linear subspace, and equip the quotient space $\mathcal{A}/V$ with the norm $\|\widehat{A}\| := \inf_{v \in V} \|A + v\|$. Show: (i) this is indeed a norm; (ii) $\mathcal{A}/V$ is complete.

(d) Show: For a Banach *-algebra $\mathcal{A}$ and a closed two-sided *-ideal $\mathcal{J} \subseteq \mathcal{A}$, the *quotient algebra* $\mathcal{A}/\mathcal{J}$ is a Banach *-algebra w.r.t. the norm from (c).

(e) Show that $\mathcal{A} := M_n(\mathbb{C})$ has only the *trivial* two-sided ideals $\{0\}$ and $\mathcal{A}$.

2.5. Polar decomposition in C^*-algebras

Show that in a unital C^*-algebra $\mathcal{A}$ every invertible element $A \in \mathcal{A}$ can be written as $A = U|A|$, where $|A| := (A^*A)^{1/2}$ and $U \in \mathcal{A}$ is unitary.

2.6. Functional calculus

Let $\mathcal{A}$ be a C^*-algebra and $A, B \in \mathcal{A}$ with $A \geq 0$ and $AB = 0$. Show that $A^{1/2}B = 0$.

(This can also be proven via functional calculus!)

2.7. Automophisms of $\mathcal{C}_0(S^1)$

For the unit circle $S^1 := \{z \in \mathbb{C} \,\big|\, |z| = 1\}$, consider the C^*-algebra $\mathcal{C}_0(S^1)$.

(a) What are the unitary elements $u \in \mathcal{C}_0(S^1)$, and what are the *inner automorphisms* π of $\mathcal{C}_0(S^1)$, i.e. the automorphisms which can be written as $\pi(f) := ufu^*$?

(b) Show that, for any $w \in S^1$, the map π_w defined by $(\pi_w(f))(z) := f(wz)$ $(\forall z \in S^1)$ is a *-automorphism of $\mathcal{C}_0(S^1)$.