1.1. Klein's inequality

(a) Let $f : (0, \infty) \to \mathbb{R}$ be a convex and differentiable function, and let $A, B \in \mathcal{B}(\mathbb{C}^d)$ be positive matrices (i.e. $A, B > 0$). Prove:

$$\mathrm{tr}\left[f(A) - f(B) - (A - B)f'(B)\right] \;\geq\; 0 \;.$$

Hint: Express A and B by their spectral decompositions $A = \sum_i a_i |e_i\rangle\langle e_i|$ and $B = \sum_j b_j |f_j\rangle\langle f_j|$, and use $\mathrm{tr}\,[X] = \sum_i \langle e_i|X|e_i\rangle$ for any $X \in \mathcal{B}(\mathbb{C}^d)$.
[In case you prefer vector notation: Let $\{e_i\}_i$ and $\{f_j\}_j$ be orthonormal eigenbases of A and B, respectively, such that $A = \sum_i a_i\, e_i e_i^*$, $B = \sum_j b_j\, f_j f_j^*$, and use $\mathrm{tr}\,[X] = \sum_i e_i^* X e_i = \sum_i (e_i, X e_i)$.]

(b) Suppose that f is *strictly convex*. When does the above inequality hold with equality?

(c) Let σ and ρ be d-dimensional quantum states (i.e. $\sigma, \rho \in \mathcal{B}(\mathbb{C}^d)$, $\sigma, \rho \geq 0$, $\mathrm{tr}\,[\rho] = \mathrm{tr}\,[\sigma] = 1$), and define the *relative entropy* $D(\sigma\|\rho) := \mathrm{tr}\,[\sigma \log \sigma - \sigma \log \rho]$.
Prove: (1) $D(\sigma\|\rho) \geq 0$. (Hint: Take $f(x) := x \log x$ above.)
(2) $D(\sigma\|\rho) = 0 \quad\Leftrightarrow\quad \sigma = \rho$.

(d) For which pairs (σ, ρ) is $D(\sigma\|\rho) = \infty$?

(e) Let $\rho_{AB} \in \mathcal{B}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ be a quantum state (on a bipartite system AB), and define the *mutual information* $I(\rho_{A:B}) := S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, where S denotes the *von Neumann entropy* $S(\rho) := -\mathrm{tr}\,[\rho \log \rho]$ (see also first lecture). Show:

$$I(\rho_{A:B}) \;=\; D(\rho_{AB}\|\rho_A \otimes \rho_B) \;.$$

Hint: $\log(\sigma \otimes \rho) = (\log \sigma) \otimes \mathbb{1}_B + \mathbb{1}_A \otimes (\log \rho)$. (Why?)

1.2. Operator monotonicity

Find two positive 2×2-matrices $A, B \in \mathcal{B}(\mathbb{C}^2)$ with $B \geq A > 0$ but $B^2 \not\geq A^2$.

1.3. Functional Analysis recap

Let $\mathcal{H}$ be a Hilbert space. Show that the space $\mathcal{B}(\mathcal{H})$ of bounded linear mappings $A : \mathcal{H} \to \mathcal{H}$, equipped with the operator norm, is a Banach space.

1.4. C^* condition

Let $\mathcal{A}$ be a Banach algebra having an involution $x \mapsto x^*$ which satisfies $\|x\|^2 \leq \|x^* x\|$ for every $x \in \mathcal{A}$. Show that $\mathcal{A}$ is a C^*-algebra.

1.5. Adjoining an identity

(a) Let $\mathcal{A}$ be a C^*-algebra (possibly without unit). Show that, for every $x \in \mathcal{A}$:

$$\|x\| \;=\; \sup\left\{ \|xy\| \;\middle|\; y \in \mathcal{A}\,,\; \|y\| \leq 1 \right\} \;.$$

(b) Compare this to the norm defined in the "adjoining an identity" from the lecture. Verify that the norm defined there makes the extended algebra *complete* and that it satisfies the C^*-condition (Hint: Bratteli/Robinson). ~~Compare the result from (a) and from the adjoining procedure also to similar expressions in subalgebras of $\mathcal{B}(\mathcal{H})$.~~

(c) Consider the spaces of complex-valued continuous functions on a topological space X that vanish at infinity, $\mathcal{C}_0(X)$. Describe $\mathcal{C}_0(X)$ more explicitly for $X = \mathbb{R}$ and for its *one-point-compactification* $X = \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. How does this correspond to the adjoining of an identity to a non-unital C^*-algebra?