SIGNAL RECOVERY & UNCERTAINTY RELATIONS

notation:

$\mathcal{B}(\mathbb{R}) :=$ Borel sets on $\mathbb{R}$

For $T \in \mathcal{B}(\mathbb{R})$ $|T| := \int_T dt$ Lebesgue measure of T

$f: \mathbb{R} \to \mathbb{C}$ signal in the time domain

$\|f\|_p := \left(\int |f(t)|^p dt \right)^{1/p}$, $L^p :=$ equivalence class of functions with $\|f\|_p < \infty$

$\hat{f}(\omega) := \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \omega t}\, dt$ Fourier transformed signal (frequency domain)
(recall Parseval: $\|\hat{f}\|_2 = \|f\|_2$)

For $A: L^p \to L^q$; $\|A\|_{p\to q} := \sup_{f \in L^p \setminus \{0\}} \frac{\|Af\|_q}{\|f\|_p} = \|A\|$ if clear from context

Def.: • For $T, W \in \mathcal{B}(\mathbb{R})$ let $P_W, P_T: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the "time-limiting" & "frequency-limiting" operators defined as $P_T f(t) := \begin{cases} f(t), & t \in T \\ 0 \end{cases}$ and
$P_W f(t) := \int_W e^{2\pi i \omega t}\, \hat{f}(\omega)\, d\omega$ (densly defined on $L^2(\mathbb{R})$)

• f is said to be "ε-L^p-concentrated" on $T \in \mathcal{B}(\mathbb{R})$ iff $\|f - P_T f\|_p \le \varepsilon \|f\|_p$

• $\hat{f}$ is "ε-L^p-concentrated /band-limited" on $W \in \mathcal{B}(\mathbb{R})$:iff $\|f - P_W f\|_p \le \varepsilon \|f\|_p$

(note that by Parseval's identity: $\|f - P_W f\|_2 = \|\hat{f} - \widehat{P_W f}\|_2$)

Lemma: $\|P_W P_T\|^2_{2\text{-}2} \le |W| \cdot |T|$

proof (sketch): note $P_W P_T f(s) = \int_W e^{2\pi i \omega s} \int_T e^{-2\pi i \omega t} f(t)\, dt\, d\omega$

$$= \int_T \int_W e^{2\pi i (s-t)\omega}\, d\omega\, f(t)\, dt$$

$$=: \int_{\mathbb{R}} k(s,t)\, f(t)\, dt \quad \Rightarrow \quad P_W P_T \text{ is compact operator}$$

$$\Rightarrow \|P_W P_T\|_{2\text{-}2} \le \underset{\substack{\uparrow \\ \text{Schatten 2-norm}}}{\|P_W P_T\|_2} = \int_T \int_W d\omega\, dt = |T| \cdot |W|$$

$\square$

Theorem: [L^2-uncertainty relation]

Let $T, W \in \mathcal{B}(\mathbb{R})$ and f and $\hat{f}$ be ε_T and ε_W L^2-concentrated on T and W respectively. Then

$$\sqrt{|W|\cdot|T|} \geq \|P_W P_T\|_{2\to 2} \geq 1-(\varepsilon_T+\varepsilon_W)$$

proof:

- $$\|f - P_W P_T f\| \underset{\substack{\uparrow \\ \Delta \text{ ineq.}}}{\leq} \|f - P_W f\| + \underbrace{\|P_W(f-P_T f)\|}_{\leq \underbrace{\|P_W\|}_{=1}\|f-P_T f\|} \leq \varepsilon_W + \varepsilon_T$$

- $$\|f - P_W P_T f\| \geq \|f\| - \|P_W P_T f\|$$

$$\Rightarrow \frac{\|P_W P_T f\|}{\|f\|} \geq 1 - \varepsilon_T - \varepsilon_W$$

$$\frac{\|P_W P_T f\|}{\|f\|} \leq \|P_W P_T\| \underset{\substack{\text{Lemma} \\ \downarrow}}{\leq} \sqrt{|W||T|}$$

$\square$

Recovery of missing segments:

- Assume $f \in L^2(\mathbb{R})$ is W-band-limited in the sense that $P_W f = f$
- Let $\eta \in L^2(\mathbb{R})$ be additive noise to f, i.e. $f \mapsto f + \eta$
- Assume the signal is missing in a time window T

$\rightarrow$ finally received signal is $\phi := (\mathbb{1} - P_T)(f+\eta)$

Thm.: If $\|P_W P_T\| < 1$ (i.e. in particular if $|W|\cdot|T| < 1$), then there is a recovery operator $R: L^2 \to L^2$ s.t.

$$\boxed{\|f - Rf\|_2 \leq \frac{\|\eta\|_2}{1 - \|P_W P_T\|}}$$

proof: Define $R := (\mathbb{1} - P_T P_W)^{-1}$ and note that $\|P_W P_T\| = \|P_T P_W\|$

Then
$$\|f - R\phi\|_2 = \|f - R(\mathbb{1} - P_T)(f + \eta)\|_2 \overset{f = P_W f}{=} \|f - f - R(\mathbb{1} - P_T)\eta\|_2 = \|R(\mathbb{1} - P_T)\eta\|_2 \leq \|R\| \underbrace{\|\mathbb{1} - P_T\|}_{=1} \|\eta\|_2$$

Moreover $\|R\| = \|(\mathbb{1} - P_T P_W)^{-1}\| \leq (1 - \|P_T P_W\|)^{-1}$, so that

$$\|f - R\phi\|_2 \leq \frac{\|\eta\|}{1 - \|P_T P_W\|}$$

□

remark: $R = (\mathbb{1} - P_T P_W)^{-1} = \sum_{k=0}^{\infty} (P_T P_W)^k$ suggests a recovery algorithm making use of alternating projections.

Thm.: [L^1-uncertainty relation] If f is ε_T-L^1 concentrated on T & bandlimited on W, then $\boxed{|W| \cdot |T| \geq 1 - \varepsilon_T}$

proof:

- by hypothesis $\frac{\|P_T f\|_1}{\|f\|_1} \geq 1 - \varepsilon_T$
- for f L^1 bandlim. it holds that $\|f\|_\infty \leq |W| \, \|f\|_1$
- on the other hand: $\|P_T f\|_1 = \int_T |f(t)| \, dt \leq \|f\|_\infty |T|$

$$\Rightarrow \frac{\|P_T f\|_1}{\|f\|_1} \leq |T| \cdot |W|$$

□

Correction of sparse noise:

Assume a band-limited signal $f = P_W f$ is sent over a "noisy channel" which adds "sparse noise" (= supported on T) so that the received signal is

$$\phi = f + P_T \eta . \qquad \text{(no bound on } \|\eta\| \text{!)}$$

With $B_1(W) := \{ f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \mid \|f\|_1 = 1 \wedge P_W f = f \}$ we get

Thm.: (Logan's phenomenon)

$$|W| \cdot |T| < \tfrac{1}{2} \Rightarrow f = \underset{\varphi \in B_1(W)}{\operatorname{argmin}} \|\varphi - \phi\|_1$$

proof: Since $|W||T_\varphi| \geq 1$ for any $\varphi \in B_1(W)$ with support T_φ

$|W||T| < \frac{1}{2}$ means that $\|P_T \varphi\|_1 < \frac{1}{2}\|\varphi\|_1$ and so

$$\|P_T \varphi\|_1 < \|P_{T^c}\varphi\|_1 \quad (\text{since } \|\varphi\|_1 = 1)$$

Therefore the best bandlimited approximation to η is zero since:

for $\varphi = P_W \varphi$:

$$\begin{aligned}
\|\eta - \varphi\|_1 &= \|P_T(\eta - \varphi)\|_1 + \|P_{T^c}(\eta - \varphi)\|_1 \\
&\underset{\Delta\text{ineq.} \,\&\, P_{T^c}\eta = 0}{\geq} \|P_T \eta\|_1 - \|P_T \varphi\|_1 + \|P_{T^c}\varphi\|_1 \\
&> \|P_T \eta\|_1 = \|\eta\|_1
\end{aligned}$$

To prove the Thm. suppose $f \neq 0$ & note that $\|f + \eta - \varphi\|_1 = \|\eta - \underbrace{(\varphi - f)}_{\text{band limited}}\|_1$

is minimized for $\varphi = f$. $\square$

note:

- T is unknown here (we just make use of small $|T|$)
- generalizations in various directions in the compressed sensing community