$\boxed{\text{Blatt 1}}$

2.)

We want to show that for $\psi, \varphi, \phi \in \mathcal{H}$ with $\|\psi\| \leq 1$, $\|\varphi\| \leq 1$, $\|\phi\| = 1$ $\phi = \lambda \varphi + (1-\lambda)\psi$ with $\lambda \in [0,1]$ implies $\varphi = \psi = \phi$.

$1 = \langle \phi, \phi \rangle = \lambda \langle \varphi, \phi \rangle + (1-\lambda) \langle \psi, \phi \rangle$

Since $|\langle \varphi, \phi \rangle| \leq 1$ and $|\langle \psi, \phi \rangle| \leq 1$, we must have $\langle \varphi, \phi \rangle = \langle \psi, \phi \rangle = 1$

Hence, Cauchy-Schwarz holds with equality both for φ, ϕ and for ψ, ϕ.

Then $\varphi = z_1 \phi$, $\psi = z_2 \phi$. $z_1 = z_2 = 1$ follows from $1 = \langle \varphi, \phi \rangle = \bar{z}_1 \langle \phi, \phi \rangle = \bar{z}_1$.

3.)

$$\langle Ux, Uy \rangle = \frac{1}{4} \sum_{i=0}^{3} i^k \| Uy + i^k Ux \|^2 \underset{\substack{\uparrow \\ U \text{ preserves norm}}}{=} \frac{1}{4} \sum_{i=0}^{3} i^k \| y + i^k Ux \|^2 = \langle x, y \rangle$$

5.)

Let $\{e_k\}$ be an ONB of $\mathcal{H}$.

Define a linear map $U : \mathcal{H} \to L_2(\mathbb{N})$, $U\psi := (\langle e_k, \psi \rangle)_{k \in \mathbb{N}}$.

Then $\| U\psi \|^2 = \sum_k |\langle e_k, \psi \rangle|^2 \underset{\substack{\uparrow \\ \text{Parseval}}}{=} \| \psi \|^2$.

So U is an isometry and consequently injective.

It is also surjective since for every $x \in L_2(\mathbb{N})$ we can define $\psi := \sum_k x_k e_k$ so that $U\psi = x$. Hence, U is bijective.