TECHNISCHE UNIVERSITÄT MÜNCHEN
Zentrum Mathematik

PROF. DR. M. WOLF **Mathematical Introduction to Quantum Information Processing** Summer 2019 Blatt 8

http://www-m5.ma.tum.de/Allgemeines/MA5057_2019S (15.06.2019)

Read the sections *Uncertainty relations* and *Information-disturbance* in the lecture notes!

1. *Commutator identity for* 2×2 *matrices.* Show that for any $A, B, C \in \mathbb{C}^{2\times 2}$ the relation $\big[[A, B]^2, C\big] = 0$ holds.

2. *Uncertainty relations.* Let $H_1, H_2 \in \mathcal{B}(\mathcal{H})$ be Hermitian, $\rho \in \mathcal{B}_1(\mathcal{H})$ a density operator and $A_i := H_i - \operatorname{tr}[\rho H_i]\,\mathbb{1}$.

   (a) Express the inequality $\operatorname{tr}[\rho B B^*] \geq 0$ as an uncertainty relation for ρ, H_1, H_2 by inserting $B := A_1 + i\gamma A_2$ and optimizing over all $\gamma \in \mathbb{R}$.

   (b) Apply the derived uncertainty relation for $\mathcal{H} \simeq \mathbb{C}^2$ to a pair of Pauli matrices. Identify 'minimal uncertainty states' that achieve equality in this uncertainty relation. Where are they located in the Bloch ball?

   (c) Which uncertainty relation is obtained when optimizing over all $\gamma \in \mathbb{C}$?

3. *Canonical commutation relation.* Let Q, P be operators on a Hilbert space $\mathcal{H}$ that satisfy the 'canonical commutation relation' $[Q, P] = i\mathbb{1}$.

   (a) Show that necessarily $\dim(H) = \infty$ and that Q, P cannot be Hilbert-Schmidt class operators.

   (b) Prove that for any $n \in \mathbb{N}$: $[Q^n, P] = inQ^{n-1}$.

   (c) Use (b) to show that Q and P cannot both be bounded operators.

4. *Tensor-power trick.* We write $A^{\otimes n} := A \otimes \ldots \otimes A$ for the n-fold tenor product of A.

   (a) Let $A, B \in \mathcal{B}(\mathcal{H})$ be Hermitian, A invertible and $A \geq \pm B$ (meaning that the inequality holds for both signs). Show that $A \geq 0$ and $A^{\otimes n} \geq \pm B^{\otimes n}$ for all $n \in \mathbb{N}$.

   (b) Show that for $\mathcal{H} \simeq \mathbb{C}^d$ there is a $\psi \in \mathcal{H}^{\otimes d}$ so that for any $A \in \mathcal{B}(\mathcal{H})$: $\det(A) = \langle \psi, A^{\otimes d}\psi \rangle$.

   (c) Use (a) and (b) to prove that in Robertson's uncertainty relation $V \geq i\sigma$ implies $\det(V) \geq \det(\sigma)$.