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Mathematical Introduction to Quantum Information Processing

Summer 2019
Blatt 5
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http://www-m5.ma.tum.de/Allgemeines/MA5057_2019S

Read the section *Composite systems and tensor products* in the lecture notes!

1. For $i \in \{1, 2\}$ consider $A_i \in \mathcal{B}(\mathcal{H}_i)$. Show that if A_1, A_2 are positive or unitary then the same holds true for $A_1 \otimes A_2$.

2. Let $\mathcal{H}_1 \simeq \mathcal{H}_2 \simeq \mathbb{C}^d$. By identifying bases of the two spaces we can define a *flip operator* $\mathbb{F} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by imposing $\mathbb{F}(\varphi \otimes \psi) = \psi \otimes \varphi$ for all φ, ψ.
   (a) Determine the eigenvalues and eigenvectors of $\mathbb{F}$.
   (b) Prove that $\mathbb{F}$ is the unique operator satisfying $\operatorname{tr}[\mathbb{F}(A \otimes B)] = \operatorname{tr}[AB]\ \forall A, B \in \mathcal{B}(\mathbb{C}^d)$.
   (c) Let $(G_i)_{i=1}^{d^2} \subset \mathcal{B}(\mathbb{C}^d)$ be any Hilbert-Schmidt-orthonormal basis of Hermitian operators. Show that $\mathbb{F} = \sum_{i=1}^{d^2} G_i \otimes G_i$.

3. Consider an element of $\mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^n)$ in block matrix representation. How can the partial traces be understood in this picture?

4. [Monogamy] Alice, Bob and Charlie share a quantum system described by a density operator $\rho \in \mathcal{B}_1(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ where $\mathcal{H}_B \simeq \mathcal{H}_C$. Suppose the reduced density operator ρ_{AB} is pure. Show that $\rho_{AC} = \rho_{AB}$ is not possible unless both are simple products (i.e. their Schmidt rank is one).

5. Denote by $\mathcal{U}_n$ all maps from $\mathcal{B}(\mathbb{C}^d)$ to itself that are of the form $\mathcal{B}(\mathbb{C}^d) \ni \rho \mapsto \sum_{i=1}^n p_i U_i \rho U_i^*$, for some $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$ and unitaries $U_i \in \mathcal{B}(\mathbb{C}^d)$. Determine an $m \in \mathbb{N}$ (as a function of d) such that $\mathcal{U}_m = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$. (Hint: you may use that in finite dimensions the convex hull of a compact set is compact.)