Read the section *Probabilistic structure of Quantum Theory* in the lecture notes!

1. For the operator norm on $\mathcal{B}(\mathcal{H})$ show that
   (a) $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{B}(\mathcal{H})$,
   (b) $\|A\| = \sup_{\|\psi\|=1} |\langle\psi, A\psi\rangle|$ for all Hermitian A.

2. Let $Q \in \mathcal{B}(\mathcal{H})$ be positive and such that $\ker(Q) = \{0\}$. Prove that $(A, B) \mapsto \mathrm{tr}\,[QA^*B]$ defines an inner product on $\mathcal{B}_2(\mathcal{H})$.

3. Construct a sequence of finite rank operators $A_n \in \mathcal{B}_0(\mathcal{H})$ that converges weakly to zero but not strongly. What about A_n^* ?

4. (a) Show that every trace-class operator can be written as a linear combination of four density operators.
   (b) Let $V \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be such that for every density operator $\rho \in \mathcal{B}_1(\mathcal{H}_1)$ the operator $V\rho V^*$ is again a density operator. What can be said about V?
   (c) Prove the Bloch ball representation of density operators on $\mathbb{C}^2$ (Hint: use the determinant).
   (d) For a given density operator on $\mathbb{C}^2$, how can one obtain the parameters in the Bloch ball representation?

5. (a) For any $\mathcal{H}$ construct a POVM that implements a 'biased coin' whose outcomes occur independently of the density operator with probabilities $\frac{1}{2}(1 \pm b)$, where $b \in [0, 1]$ is a fixed bias.
   (b) Let $M : \mathbb{B} \to \mathcal{B}(\mathbb{C}^d)$ be a sharp POVM on $(X, \mathbb{B})$. Prove that the number of pairwise disjoint elements in $\mathbb{B}$ on which M is non-zero is at most d.