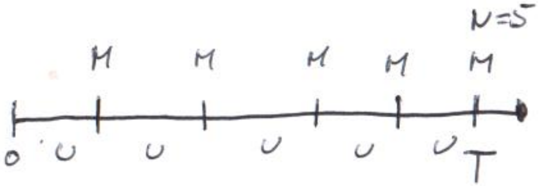
Quantum Zeno Effect.



- Measure the system N-times [redacted] until time T.
- [redacted] Initial state : $\rho_0 = |\psi_0\rangle\langle\psi_0|$, $|\psi_0\rangle \in H$, $\dim(H) < \infty$
- Schrödinger evolution : $|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$
- Projective measurement : $P_0 = |\psi_0\rangle\langle\psi_0|$, $P_1 = 1 - P_0$

  Instrument (Lüders) : $I_0(\rho) = P_0 \rho P_0$ [redacted]

  $I_1(\rho) = P_1 \rho P_1$

What will be the state at [redacted] time T, if N approaches infinity?

We calculate the probability that the out come of each measurement is [redacted] zero.

$$
\begin{aligned}
p_0([redacted]\Delta t) = tr\left(P_0 \; \rho(\Delta t)\right) &= tr\left(|\psi_0\rangle\langle\psi_0| \; |\psi(t)\rangle\langle\psi(t)| \right) \\
&= |\langle\psi_0|\psi(t)\rangle|^2 \\
&= |\langle\psi_0| e^{-iH\Delta t} \psi_0\rangle|^2 \\
&= \left| \underbrace{\langle\psi_0|\psi_0\rangle}_{=1} - i\langle\psi_0|H\psi_0\rangle \Delta t + O(\Delta t^2)\right|^2 \\
&= 1 + i\underbrace{\cancel{\langle\psi_0|H\psi_0\rangle}}_{\in\mathbb{R}}\Delta t - i\cancel{\langle\psi_0|H\psi_0\rangle}\Delta t + O(\Delta t^2)
\end{aligned}
$$

$$P_0\left((n+1)\Delta t \,\middle|\, \text{zero at } n\Delta t \right) = P_0(\Delta t) \quad , \quad \text{as} \quad \frac{\mathcal{I}_0(\rho)}{tr(\mathcal{I}_0(\rho))} = |\psi_0\rangle\langle\psi_0| \quad \text{(Post Measurement state)}$$

hence, $\quad P(\text{always outcome } 0) = \left[P_0(\Delta t)\right]^N . = P_{tot}$

So overall we have

$$P_{tot} = \left[1 + o\left(\frac{1}{N^2}\right)\right]^N = 1 + o\left(\frac{1}{N}\right) \xrightarrow[N\to\infty]{} 1$$

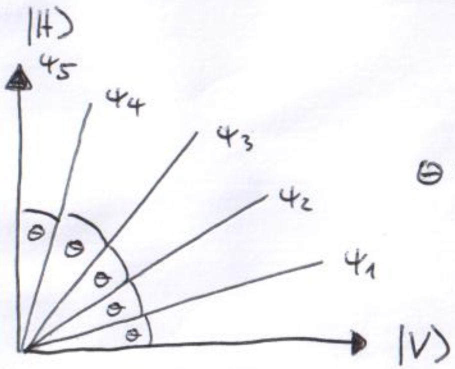
- Frequent projective measurement freezes the evolution.

Anti - Zeno Effect : (Do experiment)

- Measure system N times until time T
- Two level system : $\mathcal{H} = \text{span}(|V\rangle, |H\rangle)$ $\quad |V\rangle \perp |H\rangle$
- No internal evolution
- Initial state $\rho_0 = |V\rangle\langle V|$
- N Projective measurements $P_0^{(k)} = |\psi_k\rangle\langle\psi_k|$, $P_1^{(k)} = \mathbb{1} - P_0^{(k)}$

$$|\psi_k\rangle = R\left(\frac{k\pi}{2N}\right)|V\rangle = \cos\left(\frac{k\pi}{2N}\right)|V\rangle + \sin\left(\frac{k\pi}{2N}\right)|H\rangle$$

$k \in \{1 \dots N\}$



$\theta = \frac{\pi}{2N}$

- Lüders Instrument

Post measurement state if outcome was zero : $|\psi_k\rangle\langle\psi_k|$

Want to know the probability that we always measure zero.

$$P(\text{zero at } (k+1) \mid \text{zero at } k) = \text{tr}\left(|\psi_{k+1}\rangle\langle\psi_{k+1}| \; |\psi_k\rangle\langle\psi_k|\right) = \left|\langle\psi_{k+1}|\psi_k\rangle\right|^2$$

$$= \left|\left\langle R\left(\tfrac{(k+1)\pi}{2N}\right) V \,\middle|\, R\left(\tfrac{k\pi}{2N}\right) V\right\rangle\right|^2 = \left|\left\langle V \,\middle|\, R^*\left(\tfrac{(k+1)\pi}{2N}\right) R\left(\tfrac{k\pi}{2N}\right) V\right\rangle\right|^2$$

$$= \left|\left\langle V \,\middle|\, R\left(\tfrac{\pi}{2N}\right) V\right\rangle\right|^2 = \cos^2\left(\tfrac{\pi}{2N}\right)$$

Hence Probability for always obtaining outcome zero is

$$P(\text{always zero}) = \cos^{2N}\left(\frac{\pi}{2N}\right)$$

$$= \left(1 - \frac{\pi^2}{8N^2} + O\left(\frac{1}{N^3}\right)\right)^{2N}$$

$$= 1 - \frac{\pi^2}{4N} + O\left(\frac{1}{N^2}\right)$$

$$\xrightarrow[N\to\infty]{} 1$$

Hence the final state is $|H\rangle\langle H|$ with probability close to 1.

Interaction - Free Measurement

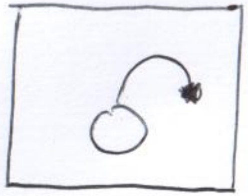
- $E(\rho) = \rho$
- In experiment: $\mathbb{1} \otimes E = \mathbb{1}$
- Single shot

$\rho_{end} = |01\rangle\langle 01|$

→ Write Probabilities

- Iterative

$\rho_{End} = |01\rangle\langle 01|$



$T(\rho) = tr(\rho)\, |0\rangle\langle 0|$

$\mathbb{1}_{B(\mathbb{C}^2)} \otimes T$

$= \frac{1}{4}$

$\rho_{End} = \cos^4\theta\, |10\rangle\langle 10|$

$= \frac{1}{4}$ $\quad + \cos^2\theta \sin^2\theta\, |01\rangle\langle 01|$

$+ \cos^3\theta \sin\theta \left(|0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| \right)$

$+ \sin^2\theta\, |00\rangle\langle 00|$

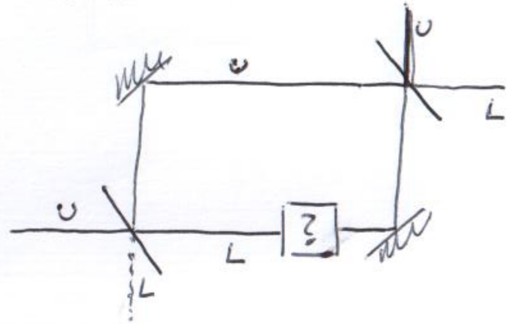
$= \frac{1}{2}$

$\rho_{End} = \cos^{2N}\theta\, |10\rangle\langle 10| + \left[1 - \cos^{2N}(\theta)\right] |00\rangle\langle 00|$

- Hilbertspace $\mathbb{C}^2 = span(|0\rangle, |1\rangle)$

no photon / one photon

- Elitzur and Vaidman



Beamsplitter: $R_\theta(\rho) = R \rho R^*$

$R|0_U 0_L\rangle = |00\rangle$

$R|11\rangle = |11\rangle$

$R|10\rangle = \cos\theta\, |10\rangle + \sin\theta\, |01\rangle$

$R|01\rangle = -\sin\theta\, |10\rangle + \cos\theta\, |01\rangle$

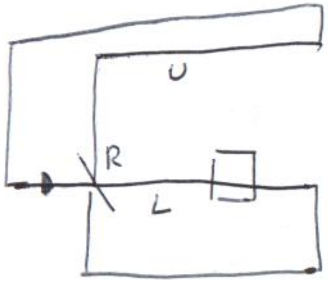
$(50/50: \theta = 45^\circ, \cos\theta = \frac{1}{\sqrt{2}})$

$(\mathbb{1} \otimes T)(R)(|10\rangle\langle 10|) =$

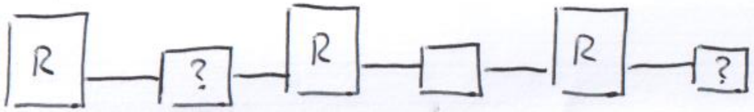
$\mathbb{1} \otimes T \big(\cos^2\theta\, |10\rangle\langle 10| + \sin\theta\cos\theta \left(|0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| \right) + \sin^2\theta\, |01\rangle\langle 01| \big)$ ⊛

$= \cos^2\theta \, |10\rangle\langle 10| + \sin^2\theta \, |00\rangle\langle 00|$

<u>Iterative</u>



$\rho_0 = |10\rangle\langle 10|$



- No bomb

$$\rho_{end} = R^N\left(|10\rangle\langle 10|\right) = |01\rangle\langle 01|$$

$$\boxed{\text{choose } \theta = \frac{\pi}{2N}}$$

- <u>With bomb</u>

Iterate (*) $\rho_{End} = \cos^{2N}\theta \left(|10\rangle\langle 10|\right) + \left[1 - \cos^{2N}(\theta)\right] |00\rangle\langle 00|$

<u>Choose</u> <u>POVM</u>

$\bar{P} = |00\rangle\langle 00| + |11\rangle\langle 11|$ $\rightarrow$ prob $1 - \cos^{2N} \rightarrow 0$

$P_B = |10\rangle\langle 10|$ $\rightarrow$ prob

$P_N = |01\rangle\langle 01|$