Problem 1: Let $T: S_1(\mathcal{H}) \to S_1(\mathcal{H})$ be a quantum channel.
Prove that there exist a countable set of operators $\{K_x \in \mathcal{B}(\mathcal{H})\}$ s.t.

(i) $\forall \varrho \in S_1(\mathcal{H}): \; T(\varrho) = \sum_x K_x \varrho K_x^*$,

(ii) $\sum_x K_x^* K_x = \mathbb{1}$.

(hint: use Stinespring's dilation theorem.)

Problem 2: Let $T: \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$ be a linear map and $\omega := \frac{1}{d} \sum_{i,j=1}^{d} |ii\rangle\langle jj|$ a maximally entangled state.

(i) Prove that T is completely positive if and only if

$$\tau := (T \otimes id_d)(\omega) \geq 0.$$

(ii) Show that the map $T \mapsto \tau$ is a bijection.

(remark: τ is called Choi-Jamiolkowski operator corresponding to T).

Problem 3 Consider a two-level atom described by a density operator $\varrho \in \mathcal{B}(\mathbb{C}^2)$ s.t. $|1\rangle\langle 1|$ and $|0\rangle\langle 0|$ are orthogonal pure states corresponding to the excited state and ground state, respectively.
Suppose that the excited state decays s.t. after time $t \in \mathbb{R}_+$ the probability of the state being still excited is $e^{-\gamma t}$ for some $\gamma > 0$.
Construct a quantum channel on $\mathcal{B}(\mathbb{C}^2)$ that is consistent with this evolution.