Hausaufgaben

5.1. Dvoretzky Dimension

a) Derive a non-trivial lower bound for the Dvoretzky dimension of l_p for $p \in (2, \infty]$.
HINT: Use the Dvoretzky criterion.

b) Show that for fixed $\epsilon > 0$ the Dvoretzky dimension of l_1^n is $k(l_1^n) = \Omega(n)$ as $n \to \infty$.

c) Let $X_n := (\mathbb{R}^n, \| \cdot \|^{(n)})$ be a sequence of metric spaces such that $k(X_n) := c(\epsilon) n^{1/4}$. Derive a non-trivial lower bound on the Dvoretsky dimension of the dual space X_n^*.

d) Formulate the geometric version of Dvoretsky's theorem. (In terms of ellipsoids, convex bodies, ...)

LÖSUNG:

a)+b) To obtain this bound, we bound the numbers m and L appearing in the bound $k < C\frac{nm^2}{2L^2}$ of the Dvoretzky criterion (see lecture 8). On exercise sheet 2, we have seen that:

$$L \leq \begin{cases} 1 & \text{if } p \geq 2 \\ n^{\frac{1}{p}-\frac{1}{2}} & \text{if } p < 2 \end{cases} .$$

To upper bound m(Note that, different from the one presented in the lecture, we use the Dvoretzky criterion in terms of the expectation value rather than the median.), observe that

$$\begin{aligned} m &= \int_{S^{n-1}} \|x\|_p d\sigma(x) \\ &\sim \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n} \|x\|_p d\gamma(x) \\ &\geq \frac{1}{\sqrt{n}} \| \int_{\mathbb{R}^n} |x| d\gamma(x) \|_p \\ &= \frac{1}{\sqrt{n}} (\sum_{i=1}^n \int_{\mathbb{R}^n} |x_i| d\gamma(x))^{1/p} \\ &= n^{\frac{1}{p}-\frac{1}{2}} \int_{\mathbb{R}^n} |x_1| d\gamma(x) \\ &= \Omega(n^{\frac{1}{p}-\frac{1}{2}}), \end{aligned}$$

where we used $\mathbb{E}(\|X\|_p) = \mathbb{E}(\||X|\|_p) \leq \|\mathbb{E}(|X|)\|_p$ is the third line. Furthermore, to get the $1/\sqrt{n}$ asymptotic behaviour used in the second line observe that:

$$\begin{aligned} \int_{\mathbb{R}^n} \|x\|_p d\gamma(x) &= C \int_0^\infty r^{n-1} e^{-r^2} \int_{S^{n-1}} \|rx\|_p d\sigma(x) \\ &= C \int_0^\infty r^n e^{-r^2} \int_{S^{n-1}} \|x\|_p d\sigma(x), \end{aligned}$$

where C is just some constant. Now we just have to prove that $C\int_0^\infty r^n e^{-r^2} \sim 1\sqrt{n}$. To see this, we choose $p = 2$. Then,

$$C\int_0^\infty r^n e^{-r^2} = \int_{\mathbb{R}^n} \|x\|_2 d\gamma(x).$$

Hence it suffices to prove that $\int_{\mathbb{R}^n} \|x\|_2 d\gamma(x) \sim \sqrt{n}$. We do this by bounding the second and the fourth moment of $\|x\|_2$:

$$\begin{aligned}\int_{\mathbb{R}^n} \|x\|_2^2 d\gamma(x) &= \sum_{i=1}^n \int_{\mathbb{R}} x_i^2 d\gamma(x_i)\\ &= n\int_{\mathbb{R}} x_1^2 d\gamma(x_1) = \Omega(n).\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^n} \|x\|_2^4 d\gamma(x) &= \int_{\mathbb{R}^n} \sum_{i,j=1}^n x_i^2 x_j^2 d\gamma(x)\\ &= \sum_{i\neq j=1}^n \int_{\mathbb{R}} x_i^2 d\gamma(x_i) \int_{\mathbb{R}} x_j^2 dx_j d\gamma(x_j) + \sum_{i=1}^n \int_{\mathbb{R}} x_i^4 d\gamma(x_i)\\ &= \Omega(n^2).\end{aligned}$$

Define the random variable $Z : S^{n-1} \to \mathbb{R}$ by $Z(x) := \|x\|_2$. Then the above computation shows that there is a constant $D \in \mathbb{R}$ such that $C\mathbb{E}(Z^2)^{1/2} \geq \mathbb{E}(Z^4)^{1/4}$. But then, using Hoelder with $p = 1/3$ and $q = 2/3$, we find:

$$\mathbb{E}(Z^2) = \mathbb{E}(Z^{2/3}Z^{4/3}) \leq \mathbb{E}(Z)^{2/3}\mathbb{E}(Z^4)^{1/3} \leq \mathbb{E}(Z)^{2/3}D^{1/3}\mathbb{E}(Z^2)^{2/3}.$$

Thus we find that

$$\Omega(n) = \mathbb{E}(Z^2) \leq D\mathbb{E}(Z)^2.$$

Additionally we have $\mathbb{E}(Z)^2 \leq \mathbb{E}(Z^2)$ by Jensen inequality and this proves the claim. Finally, we obtain:

$$k(l_p^n) \geq \begin{cases} \Omega(n^{2/p} & \text{if } p \geq 2 \\ \Omega(n) & \text{if } p < 2 \end{cases}.$$

But obviously $k(l_p^n) \leq \Omega(n)$ and this proves b).

c) In the lecture we have seen that $d(X, l_2^n) \leq \sqrt{n}$ for all $X \simeq \mathbb{R}^n$. Then, using the inequality from the lecture, we find

$$k(X_n)k(X_n^*) \geq \frac{cn^2}{d(X, l_2^n)^2} \geq cn.$$

Thus, we find $k(X_n^*) \geq C'n^{1-1/4}$.

d) For every $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that the following holds: If $X := (\mathbb{R}^n, \|\cdot\|)$ is a normed space with unit ball B_X, then there exists an ellipsoid $\mathcal{E}_0 \subset \mathbb{R}^n$ and a subspace $V \subseteq \mathbb{R}^n$ such that:

- There exists $r > 0$ for which

$$r(\mathcal{E}_0 \cap V) \subseteq B_X \cap V \subseteq r(1+\epsilon)(\mathcal{E}_0 \cap V)$$

- $\dim V \geq C(\epsilon)\log n$.

5.2. Dvoretzky-Rogers Theorem

Using the Dvoretzky-Rogers lemma from the lecture proof the Dvoretzky-Rogers theorem:

If a Banach space X is such that unconditional summability implies absolute summability, then $\dim X < \infty$.

LÖSUNG:

We will prove the contrapositive. Assuming that $\dim X = \infty$, we construct an unconditionally convergent series that is not absolutely convergent:

Take any $(a_i)_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} a_i^2 < 1$. Find a sequence let $n_0 < n_2 < ...$ such that $\sum_{i=n_k}^{\infty} a_i \leq 2^{-2k}$.

Now let Y_k be a $2(n_{k+1} - n_k) + 1$ subspace of X. By the Dvoretzky-Rogers Lemma we can find vectors $\{y_i\}_{i=n_k}^{n_{k+1}-1}$ with $\|y_i\| \geq 1/2$ such that for all λ_i and $n_k < l \leq n_{k+1}$:

$$\sqrt{\sum_{i=n_k}^{l} \lambda_i^2} = \sum_{i=n_k}^{l} \|\lambda_i y_i\|_{\mathcal{E}} \geq \sum_{i=n_k}^{l} \|\lambda_i y_i\| \geq \|\sum_{i=n_k}^{l} \lambda_i y_i\|.$$

Here $\|\cdot\|_{\mathcal{E}}$ denotes the norm associated to the maximum volume ellipsoid contained in B_X.

Repeating the construction for all k yields a sequence $(y_n)_{n\in\mathbb{N}}$.

Now let $x_i := a_i \frac{y_i}{\|y_i\|}$ and let $\epsilon_i = \pm$. Then, for $n, m > n_k$,

$$\|\sum_{i=n}^{m} \epsilon_i x_i\| \leq \sqrt{\sum_{i=n_k}^{n_{k+1}-1} \frac{a_i^2}{\|y_i\|^2}} + \sqrt{\sum_{i=n_{k+1}}^{n_{k+2}-1} \frac{a_i^2}{\|y_i\|^2}} + ... \leq 2\sqrt{\sum_{i=k}^{\infty} 2^{-k}}.$$

I.e. $(\sum_{i=1}^{n} \epsilon_i x_i)_n$ is a Cauchy sequence and thus $(x_n)_n$ is unconditionally summable. Now choosing $\|x_i\| = a_i = C/i$, we have $\sum_{i=1}^{\infty} a_i^2 < 1$ for an appropriately choosen C but $(x_n)_n$ is clearly not absolutely summable since $\sum_{i=1}^{\infty} \|x_i\| = \sum_{i=1}^{\infty} a_i = \infty$.