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Asymptotic Geometric Analysis

http://www-m5.ma.tum.de/Allgemeines/MA5024_2015W

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Lösungsblatt 2
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Hausaufgaben

2.1. Haar Measure

Let $L \in G_{n,k}$ be a k-dimensional subspace of $\mathbb{R}^n$ and let $U \in O(n)$ be a Haar-random rotation. Show that UL is a Haar-random k-dimensional subspace.

Lösung:

Let $A \subseteq G_{n,k}$ be a Borel subset and let $V \in O(n)$. Then,

$$\begin{aligned}
&P(\{UL \in VA\}) = \sigma(\{U \in O(n) : UL \in VA\}) = \sigma(\{U \in O(n) : V^{-1}UL \in A\}) \\
&= \sigma(\{VU \in O(n) : UL \in A\}) = \sigma(V\{U \in O(n) : UL \in A\}) = \sigma(\{U \in O(n) : UL \in A\}) \\
&= P(\{UL \in A\}).
\end{aligned}$$

The statement then follows form the uniqueness of the Haar measure.

2.2. Concentration of Measure

a) Let $A := \{x \in S^{n-1} : x \leq 0\}$ be the southern hemisphere and $A_\delta := \{x \in S^{n-1} : \exists y \in A : \|x - y\| \leq \delta\}$ be the complement of a 'spherical cap' for $\delta \in [0, 1]$. Provide an elementary provable bound on the measure $1 - \sigma(A_\delta)$ of the spherical cap.

b) Argue that this implies concentration of measure around any set B with $\sigma(B) = 1/2$, if we assume the isoperimetric theorem for the sphere: Among all sets of fixed measure on S^{n-1}, the spherical cap has the smallest boundary measure.

c) Let $f : S^{n-1} \to \mathbb{R}$ be 1-Lipschitz and $m := \mathrm{med}(f)$. Prove an upper bound for $|\mathbb{E}(f^2) - m^2|$ in terms of n.

d) Let $f : \mathbb{R}^n \to \mathbb{R}$ be 1-Lipschitz and γ the Gaussian measure on $\mathbb{R}^n$. Derive a lower bound for $\gamma(|f - \mathrm{med}(f)| \leq \delta)$ in terms of δ.
HINT: Use the geometric Gaussian measure concentration result stated in the lecture.

Lösung:

a) Let ϵ, θ, r be as defined in the picture and note that:

$$\sin(\theta) = \frac{\epsilon}{1}$$
$$\sin(\theta/2) = \frac{\delta/2}{1}.$$

Furthermore note that for $\theta \in [0, \pi/2]$

$$\begin{aligned}
\frac{\epsilon}{\delta} &= \frac{\sin(\theta)}{2\sin(\theta/2)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin(\theta/2)} \\
&= \cos(\theta/2) \leq \cos(\pi/4) = \frac{1}{\sqrt{(2)}}.
\end{aligned}$$

Let $C = \text{conv}(\{0\} \cup B)$, where B is the polar cap. We then find

$$\sigma(B) = \frac{\text{vol}(C)}{\text{vol}(B_{l_2^n})} \leq \frac{\text{vol}(B_{l_2^n}(r)}{\text{vol}(B_{l_2^n})}$$
$$= \frac{r^n \text{vol}(B_{l_2^n}}{\text{vol}(B_{l_2^n})} = r^n = (1-\epsilon^2)^{n/2} \leq e^{-n\epsilon^2/2} \leq e^{-n\delta^2/2},$$

where the first inequality certainly holds for small enough δ.

b) The surface area of a measurable set $B \subseteq S^{n-1}$ is defined as $S(B) := \lim_{\epsilon \to 0} \frac{1}{\epsilon}(\text{vol}(B_\epsilon) - \text{vol}(B))$. Thus it gives the rate of change of volume when the set is increased. Since for sets of fixed measure this rate is the smallest for the polar cap by the isoperimetric theorem and thus the volume grows stronger for all sets different from the polar cap. In particular the concentration of measure holds for all sets.

c) Note that $g(x) := |f - m|^2$ and note that $g(x) = 2\int_0^\infty t\chi(x,t)dt$ where

$$\chi(x,t) := \begin{cases} 1 & \text{if } t \leq g(x) \\ 0 & \text{if } else \end{cases}$$

Then we find using Fubinis theorem

$$\mathbb{E}(|f-m|^2) := \int_{S^{n-1}} g(x)d\sigma(x) = \int_{S^{n-1}} (2\int_0^\infty t\chi(x,t)dt)d\sigma(x)$$
$$= 2\int_0^\infty tP(|f-m| > t)dt \leq 2\int_0^\infty 4te^{-nt^2/4}dt = 16/n.$$

Hence we find

$$|\mathbb{E}(f)^2 - m^2| = \mathbb{E}(|f-m|^2) + 2m\mathbb{E}(|f-m|) \leq 16/n + 8\sqrt{\pi}/\sqrt{n}.$$

Where we used the bound on $\mathbb{E}(|f-m|)$ from the lecture.

d) Let $A := \{x \in S^{n-1} : f(x) \leq m\}$. Then $\gamma(A) \geq 1/2$. Furthermore, if $y \in A_\delta$, then $\exists x \in A$ such that $\delta \geq \|x-y\|_2 \geq |f(x) - f(y)|$. Hence $y \in A_\delta$ implies $f(y) \leq med(f) + \delta$. Thus we deduce from the concentration of measure bound from the lecture

$$\gamma[f \geq med(f) + \delta] \leq 1 - \sigma(A_\delta) \leq 2e^{-\delta^2/4}.$$

The other inequality is proven similarly. We hence find

$$\gamma[|f - med(f)| < \delta] \geq 1 - 4e^{-\delta^2/4}.$$

2.3. Johnson-Lindenstrauss

a) What can be said about the preservation of angles in the Johnson-Lindenstrauss theorem?

b) Let $\{u_1, ..., u_n\} \subseteq \mathbb{R}^n$ be an orthonormal basis. Show that for any $\epsilon \in (0,1)$ there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^k$ with $k = O(\log(n)/\epsilon^2)$ s.t. $|\langle L(u_i), L(u_j)\rangle| \leq \epsilon\|L(u_i)\|\|L(u_j)\|$ for all $i \neq j$.

LÖSUNG:

a) Angles can be preserved! Assume we start with a set of points C with $|C| = n$. Note that for $u, v \in \mathbb{R}^n$

$$\cos(\phi)\|u\|\|v\| := \langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2). \tag{1}$$

We can ensure that both $\|u+v\|^2$ and $\|u-v\|^2$ are preserved by adding for every point $v \in C$ the point $-v$ to C and additionally adding the origin. In this way we enlarge the set C to a set of cardinality at most $2n+1$.

b) Assume w.l.o.g. that $\|Lu_i + Lu_j\| \geq \|Lu_i - Lu_j\|$. We then find

$$\begin{aligned}
\frac{\langle Lu_i, Lu_j \rangle}{\|Lu_i\|\|Lu_j\|} &= \frac{1/4(\|Lu_i + Lu_j\|^2 - \|Lu_i - Lu_j\|^2)}{\|Lu_i\|\|Lu_j\|} \\
&\leq \frac{1/4((1+\epsilon^2)\|u_i + u_j\|^2 - (1-\epsilon)^2\|u_i - u_j\|^2)}{(1-\epsilon)\|u_i\|(1-\epsilon)\|u_j\|} \\
&= (1-\epsilon)^{-2}\left(\langle u_i, u_j \rangle + \epsilon/2(\|u_i + u_j\|^2 + \|u_i + u_j\|^2)\right) \\
&= (1-\epsilon)^{-2}(0 + \epsilon/2 \cdot 4) = \frac{2\epsilon}{(1-\epsilon)^2}.
\end{aligned}$$

This proves the claim for small enough ϵ.