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Asymptotic Geometric Analysis

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Hausaufgaben

2.1. Banach-Mazur distance

a) Show that the geometric and analytic version of the Banach-Mazur Distance coincide. That is, if $B_X,\ B_Y$ are unit balls of normed spaces $X \simeq Y \simeq \mathbb{R}^n$, then $d(B_X, B_Y) = d(X, Y)$.

b) Show that $X \simeq Y \simeq \mathbb{R}^n$ are isometrically isomorphic iff $d(X, Y) = 1$.

c) Let $1 \le p \le q \le 2$ or $2 \le p \le q \le \infty$. Show that for every $n \in \mathbb{N}$:

$$d(l_p^n, l_q^n) = n^{1/p-1/q}.$$

HINT: For " $\ge$ " use $d(l_2^n, l_\infty^n) = \sqrt{n}$ together with the submultiplicativity of the Banach-Mazur distance.

LÖSUNG:

a) If $d(X, Y) = c$ then there exists $T \in \mathcal{B}(X, Y)$ such that $\|T\|\|T^{-1}\| = c$ [1]. Then $TB_X \subseteq \|T\|B_Y$ [2] and $T^{-1}B_Y \subseteq \|T^{-1}\|B_X$ imply

$$\frac{1}{\|T^-1\|}B_Y \subseteq TB_X \subseteq \|T\|B_Y.$$

Hence $d(B_X, B_Y) \le \|T\|\|T^{-1}\| = c = d(X, Y)$.
Conversely, suppose $\frac{1}{b}B_Y \subseteq TB_X \subseteq aB_Y$ for some $T \in \mathcal{B}(X, Y),\ a, b \in \mathbb{R}$. Then $\|T\| = \sup_{x\in B_X} \|Tx\|_Y = \sup_{x\in TB_X} \|x\|_Y \le \sup_{x\in aB_Y} \|x\|_Y = a$ and similarly $\|T^{-1}\| = \sup_{x\in B_Y} \|T^{-1}x\|_X = \sup_{x\in T^{-1}B_Y} \|x\|_X \le \sup_{x\in bB_X} \|x\|_X \le b$ so that $\|T\|\|T^{-1}\| \le ab$.

b) Note that $1 = d(X, Y) = \inf\{\|T\|\|T^{-1}\| : T \in \mathcal{B}(X, Y)\} = \inf\{\|T^{-1}\| : T \in \mathcal{B}(X, Y),\ \|T\| = 1\}$. By compactness of $\{T \in \mathcal{B}(X, Y) : \|T\| = 1,\ \|T^-1\| \ge C\}$, $C \in \mathbb{R}_+$, the infimum is attained, i.e. there exists $T \in \mathcal{B}(X, Y)$ with $\|T\| = \|T^{-1}\| = 1$.

c) By duality we can assume w.l.o.g that $2 \le q \le p \le \infty$. Then $\|x\|_q \le \|x\|_p \le n^{1/p-1/q}\|x\|_q$ using Hölder inequality: Let $x \in \mathbb{R}^n_+$, then $\|x\|_p^p = \sum_{i=1}^n x_i^p \cdot 1 = \langle x^p, e\rangle \le \|x^p\|_r\|e\|_s = n^{1/s}\|x^p\|_r = n^{\frac{r-1}{r}}\|x^p\|_r$, where $e := (1, 1, ..., 1)$, $x^p := (x_1^p, x_2^p, ..., x_n^p)$ and $1/r + 1/s = 1$. Noting that $\|x^p\|_r^{1/p} = \|x\|_{rp}$ we conclude that $\|x\|_p \le n^{\frac{r-1}{pr}}\|x\|_{pr}$ for all $p, r > 1$ and thus $\|x\|_p \le n^{\frac{1}{p}-\frac{1}{q}}\|x\|_{pr}$ for all $1 < p \le q$.
Conversely,

$$\sqrt{n} \le d(l_2^n, l_\infty^n) \le d(l_2^n, l_p^n)d(l_p^n, l_q^n)d(l_q^n, l_\infty^n) \le n^{1/2-1/p}d(l_p^n, l_q^n)n^{1/q}.$$

2.2. Brunn-Minkovski inequality

[1] Note that the Banach-Mazur distance is attained. This is because the set of $\{T \in \mathcal{B}(X, Y) : \|T\| = 1,\ \|T^{-1}\| > C\}$ is compact for given $C \in \mathbb{R}_+$ by the continuity of the inversion.

[2] Note that $\sup_{x\in TB_X} \|x\|_Y = \sup_{x\in B_X} \|Tx\|_Y = \|T\|$.

a) The Prékopa-Leidler inequality states that $\|h\|_1 \geq \|f\|_1^{1-\lambda}\|g\|_1^{\lambda}$ whenever $h, f, g : \mathbb{R}^n \to \mathbb{R}^+$ are Lebesgue integrable and for $\lambda \in (0,1)$ statisfy:

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(x)^{\lambda}.$$

Prove the Brunn-Minkovski inequality from the Prékopa-Leidler inequality.

b) For a Lebesgue measurable set $K \in \mathbb{R}^n$ define the centroid of K by

$$x(K) := \frac{1}{vol(K)} \int_K y dy.$$

Let $C := \{x \in \mathbb{R}^n : \ x_1 \in [0,h], \sum_{i=1}^{n-1} x_i^2 \leq x_1^2\}$ be the standard cone of hight h. Let $\overline{x}$ be the centroid of C. Let $L := C \cap \{x \in \mathbb{R}^n : x_1 \leq \overline{x}_1\}$. Show that $\frac{1}{2} \geq \frac{vol(L)}{vol(C)} \geq \frac{1}{e}$.

c) Prove Grünbaum's Theorem: Let $K \subseteq \mathbb{R}^n$ be a convex body and divide K into $K_1, \ K_2$ using a hyperplane. If K_1 contains the centroid of K, then

$$\frac{vol(K_1)}{vol(K)} \geq 1/e.$$

HINT: Assume w.l.o.g. that the centroid lies in the origin and that the hyperplane is given by $\{x \in \mathbb{R}^n : x_1 = 0\}$.

- Construct a set K' as follows. Replace each slice $K_t = K \cap \{x \in \mathbb{R}^n : x_1 = t\}$ with a ball of the same volume. Show that K' is convex using the Brunn-Minkovski inequality.
- Show that, without increasing the ratio, K' can be deformed into a cone and use c).

LÖSUNG:

a) For $A, B \subseteq \mathbb{R}^n$ non-empty and compact , set $f = \chi_A, \ g = \chi_B$ and $h = \chi_{(1-\lambda)A + \lambda B}$. First note that $\|f\|_1 = vol(A)$ and similarly for g, h. Furthermore, note that if $x \in A$ and $y \in B$, then $(1-\lambda)x + \lambda y \in (1-\lambda)A + \lambda B$. Thus,

$$h((1-\lambda)x + \lambda y) \geq f(x)g(y) = f(x)^{1-\lambda} g(x)^{\lambda}.$$

We then conclude from the The Prékopa-Leidler inequality

$$vol((1-\lambda)A + \lambda B) = \|h\|_1 \geq \|f\|_1^{1-\lambda}\|g\|_1^{\lambda} = vol(A)^{1-\lambda} vol(B)^{\lambda}.$$

b) By symmetry, the only non-vanishing component of $\overline{x}$ is

$$\overline{x}_1 = \frac{1}{vol(C)} \int_C y_1 dy = \frac{\int_0^h t^n dt \ vol(S^n)}{\int_0^h t^{n-1} dt \ vol(S^n)} = \frac{n}{n+1} h.$$

We then find

$$\frac{1}{2} \geq \frac{vol(L)}{vol(C)} = \left(\frac{n}{n+1}\right)^n \to 1/e.$$

Since $(\frac{n}{n+1})^n$ is monotone this proves the claim.

c) Let $K_t' := K' \cap \{x \in \mathbb{R}^n : x_1 = t\}$. The radius of K_t' is proportional to $vol(K_t)^{n-1}$. Note that $(1-\lambda)K_{t_1} + \lambda K_{t_2}) \subseteq K_{(1-\lambda)t_1 + \lambda t_2}$ by convexity of K. Hence, from the Brunn-Minkovski inequality, we deduce that

$$vol(K_{(1-\lambda)t_1+\lambda t_2})^{n-1} \geq vol((1-\lambda)K_{t_1} + \lambda K_{t_2})^{n-1} \geq (1-\lambda) vol(K_{t_1})^{n-1} + \lambda vol(K_{t_2})^{n-1}.$$

Hence $vol(K_t)^{n-1}$ is concave in t and thus K' is convex.

Let us now deform K' into a cone: Let $K'_+ := \bigcup_{t \geq 0} K'_t$ and $K'_+ := \bigcup_{t<0} K'_t$. Let C be the cone with base K'_0 and vertex y on the positive x_1-axis such that $vol(C) = vol(K'_+)$ and extend C to a cone C' such that $vol(C') = vol(K)$. Then, by concavity of of the radius function the centroid q of C' is contained in K'_+. Finally,

$$\frac{K_1}{K} \geq \frac{C' \cap \{x_1 \geq q_1\}}{C'} \geq 1/e$$

where we used c).