TECHNISCHE UNIVERSITÄT MÜNCHEN
Zentrum Mathematik

Prof. Dr. M. Wolf **Mathematical Foundations of Machine Learning** Summer 2016 Blatt 7

http://www-m5.ma.tum.de/Allgemeines/MA4801_2016S (09.06.2016)

The following homework will be discussed on June 15th to 17th:

I. Neural network with infinite VC-dimension

Define $\sigma_c(z) := \frac{1}{1+e^{-z}} + cz^3 e^{-z^2} \sin z$ with $c \geq 0$ so that σ_0 is the logistic sigmoid function.

1. Make a qualitative comparison of the graphs of the functions σ_0 and σ_c for small but non-zero values of c.

2. Consider a feedforward neural network with one input, a single output neuron whose activation function is $z \mapsto \operatorname{sgn}(z)$ and a single hidden layer with two neurons using σ_c with $c > 0$ as activation function. Show that the class of functions represented by this architecture has infinite VC-dimension (hint: use what you know about the VC-dimension of the function class given by $\mathbb{R}_+ \ni z \mapsto \operatorname{sgn}\sin(z\alpha)$ with $\alpha \in \mathbb{R}$).

II. Neural networks - geometric interpretation

Consider a feedforward neural network with activation functions $\sigma(z) = \mathbb{1}_{z \geq 0}$ and d inputs so that it represents a function of the form $f : \mathbb{R}^d \to \{0,1\}$. Define $A := f^{-1}(\{1\}) \subseteq \mathbb{R}^d$.

1. Assume the network has a single hidden layer with k neurons. What can you say about A? (hint: recall which geometric objects correspond to single Perceptrons)

2. Assume $d = 2$. Find a neural network architecture for which A can be the TUM logo.

III. Neural networks - learning complexity

Consider a feedforward neural network with d inputs, a single hidden layer with three neurons and a single output neuron. Assume all activation functions are $\sigma(z) = \mathbb{1}_{z \geq 0}$ and that the output neuron has all weights and the threshold fixed so that it acts as $x \mapsto \sigma(\sum_{i=1}^{3}(x_i - 1))$. Regarding the weights and threshold values of the three hidden neurons as free parameters gives rise to a function class $\mathcal{F}_d \subseteq \{0,1\}^{\mathbb{R}^d}$.

1. If $f \in \mathcal{F}_d$, what is the geometry of the set $f^{-1}(\{1\})$?

2. Consider a graph $G = (V, E)$. Look up (in the online or off-line source of your choice) what the 3-coloring problem is for G. Look up what is known about its computational complexity.

3. For any graph $G = (V, E)$ define a 'training data set' $S \in \left(\{0,1\}^{|V|} \times \{0,1\}\right)^n$ with $n = |V| + |E| + 1$ as follows: denote by $e_i \in \mathbb{R}^{|V|}$ the unit vector that has component 1 at the i'th position. Then S is assumed to contain the elements $(e_i, 0)$ for all $i = 1, \ldots, |V|$, the elements $(e_i + e_j, 1)$ for each edge $(i,j) \in E$ and the element $(0, 1)$. Show that G is 3-colorable iff there is a function $h \in \mathcal{F}_{|V|}$ that correctly classifies S.

4. What does this imply for the computational complexity of ERM for neural networks?