Lecture 3
Complete metric spaces

1 Complete metric spaces

1.1 Definition. Let (X, d) be a metric space. We say that a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a *Cauchy sequence* if for all $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq N_\varepsilon$, $d(x_n, x_m) < \varepsilon$, or in short (and not completely precisely), if

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0.$$

1.2 Remark. Every convergent sequence is Cauchy, but the converse is not true.

1.3 Definition. We say that a metric space (X, d) is *complete* if every Cauchy sequence in X has a limit in X, i.e., every Cauchy sequence is convergent.

1.4 Example. $\mathbb{R}^n$ with the Euclidean metric is complete.

One may then ask whether all metric spaces are complete. The answer is not; see e.g. the homework exercises for the first week. We have the following more general statement:

1.5 Theorem. Let (X, d) be a complete metric space and $S \subseteq X$. Then S is complete if and only if S is closed.

Proof. ($\Longrightarrow$) Let $x \in \overline{S}$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq S$ converging to x. Obviously, this sequence is a Cauchy sequence, and, since S is complete, it converges to some $\tilde{x} \in S$. Since the limit of a sequence is unique in a metric space, we see that $x = \tilde{x} \in S$.

($\Longleftarrow$) Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in S. Since X is complete, $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in X$. But as S is closed, x has to be in S. □

1.6 Remark. The ($\Longrightarrow$) part of the above proof actually shows that a complete subspace of any (not necessarily complete) metric space is closed.

1.7 Remark. Note the analogy with the statement that a subset in a compact Hausdorff space is compact if and only if it is closed.

1.8 Remark. The above theorem provides a way to prove completeness of a metric space by considering it as a subspace of a larger complete metric space, and proving closedness.

2 Completion of metric spaces

2.1 Definition. (isometries) Let (X_1, d_1) and (X_2, d_2) be metric spaces. We say that a map $f : X_1 \to X_2$ is an *isometry* if

$$d(f(x), f(y)) = d(x, y), \qquad x, y \in X_1.$$

Note that every isometry f is automatically injective, and its inverse is also an isometry on the range of f.

2.2 Definition. Let (X_1, d_1) and (X_2, d_2) be metric spaces. We say that X_1 and X_2 are *isometric*, if there exists a bijective isometry $f : X_1 \to X_2$.

Isometric metric spaces are equivalent for all properties that only concern their metric.

2.3 Remark. Note that a bijective isometry between metric spaces is also a homeomorphism (check!), and hence isometric metric spaces are topologically equivalent. The converse is not true; the existence of a homeomorphism between two metric spaces doesn't imply the existence of an isomorphism. In particular, it is possible that two metric spaces are homeomorphic but one is complete while the other is not; see, e.g., the example from the first exercise class, with the metric $d(x, y) := |\arctan x - \arctan y|$ on $\mathbb{R}$, which is topologically equivalent to the Euclidean metric, but not complete.

Completeness plays a key role in many important theorems in (functional) analysis, as we will see repeatedly in this course. The following theorem tells that while not every metric space is complete, every metric space can be considered as a dense subspace of a complete metric space.

2.4 Definition. Let (X, d_X) be a metric space. We say that a metric space (Y, d_Y) is a *completion* of X, if there exists an isometry $f : X \to Y$ such that $f(X)$ is dense in Y, i.e., $\overline{f(X)} = Y$.

2.5 Theorem. Every metric space has a completion.

Proof. (Sketch) Let (X, d) be a metric space. Let Y_0 be the set of Cauchy sequences in X, and introduce the following relation on Y_0:

$$(x_n) = (x_n)_{n\in\mathbb{N}} \sim (y_n) = (y_n)_{n\in\mathbb{N}} \quad \text{if} \quad \lim_{n\to+\infty} d(x_n, y_n) = 0.$$

One can easily see that this is an equivalence relation, and we denote the equivalence class of a sequence (x_n) by $[(x_n)]$. We define Y to be the set of equivalence classes of this relation, i.e.,

$$Y := \{\text{Cauchy sequences}\} \big/ \sim \, = \{[(x_n)] : (x_n) \text{ is a Cauchy sequence}\}$$

One can easily see that if $(x_n), (y_n)$ are Cauchy sequences then the sequence $d(x_n, y_n)$, $n \in \mathbb{N}$, is convergent, and the limit only depends on the equivalence classes of (x_n) and (y_n). We introduce a metric d_Y on Y by

$$d_Y\left([(x_n)], [(y_n)]\right) := \lim_{n\to+\infty} d(x_n, y_n).$$

One can easily see that this is indeed a metric. Moreover, (Y, d_Y) is complete (we omit the proof of this here). For every $x \in X$, let $f(x)$ be the equivalence class of the constant sequence $x_n = x$, $n \in \mathbb{N}$. Then f is an isometric embedding of X into Y, as

$$d_Y(f(x), f(y)) = \lim_{n\to+\infty} d(x, y) = d(x, y), \qquad x, y \in X.$$

Finally, we have to prove that $f(X)$ is dense in Y. Let $[(y_n)] \in Y$, and define $x^{(m)} := y_m$. Then

$$d_Y\left(f(x^{(m)}), [(y_n)]\right) = \lim_{n\to+\infty} d(x^{(m)}, y_n) = \lim_{n\to+\infty} d(y_m, y_n),$$

and

$$\lim_{m\to+\infty} d_Y\left(f(x^{(m)}), [(y_n)]\right) = \lim_{m\to+\infty} \lim_{n\to+\infty} d(y_m, y_n) = 0$$

since (y_n) is a Cauchy sequence. □

2.6 Remark. The completion of a metric space is unique up to isomorphism. More precisely, if (Y_1, d_1) and (Y_2, d_2) are complete metric spaces, and $f_i : X \to Y_i$ is an isometric embedding of X into a dense subspace of Y_i for $i = 1, 2$, then $g := f_2 \circ f_1^{-1} : f(X_1) \to f(X_2)$ is an isometry that maps $f_1(X)$ into $f_2(X)$ such that $g(f_1(x)) = f_2(x)$, $x \in X$, and it has a unique extension to an isometric bijection between Y_1 and Y_2.

According to the above Remark, we can talk about "the" completion of a metric space.

2.7 Remark. If (X, d) is a metric space and $X_0 \subseteq X$ is dense in X then X is a completion of X_0 with the trivial embedding $f(x) := x$, $x \in X_0$.

2.8 Example. Let $X := C([0,1], \mathbb{R}) := \{f : [0,1] \to \mathbb{R} \text{ continuous}\}$ be equipped with the norm $\|f\|_\infty := \max_{t\in[0,1]} |f(t)|$. We have seen in the exercise class that this is a Banach space, i.e., it is a complete metric space with the metric $d(f,g) := \|f - g\|_\infty$. Let X_0 be the set of polynomial functions on $[0,1]$. By the Weierstrass approximation theorem, X_0 is dense in X, and hence, by the above Remark, $C([0,1], \mathbb{R})$ is the completion of the set of polynomial functions on $[0,1]$ (w.r.t. the maximum norm $\| \ \|_\infty$).

3 The Baire category theorem

Introduction: Consider a probability space $(\mathcal{X}, \mathcal{S}, P)$, where $\mathcal{X}$ is a set, $\mathcal{S}$ is a sigma-field (or sigma-algebra), and P is a probability measure. Then, if $A_n \in \mathcal{S}$, $n \in \mathbb{N}$, is a countable family of measurable sets so that $P(A_n) = 1$, $n \in \mathbb{N}$, then $P\left(\cap_{n\in\mathbb{N}} A_n\right) = 1$. That is, the countable intersection of sets that are "almost the whole space", is a set that is again "almost the whole space".

The analogous statement in topology is the so-called Baire category theorem. Recall that a set M in a topological space is dense if its closure is the whole space. This in some sense means that a dense set M is "almost the whole space"; for instance, in a metric space it means that we can approximate all points of the space by sequences in M. However, density is not enough for an analogue of the probability theoretic example above. Indeed, if we take our space to be $[0,1]$ with the usual Euclidean topology, and choose M_1 to be the rational numbers in $[0,1]$, and M_2 to be the irrational numbers in $[0,1]$, then both M_1 and M_2 are dense, but $M_1 \cap M_2 = \emptyset$. However, we have the following:

3.1 Theorem. (Baire category theorem) Let (X, d) be a complete metric space, and let U_n, $n \in \mathbb{N}$, be a countable family of dense open sets in X. Then $\cap_{n\in\mathbb{N}} U_n$ is dense in X.

Proof. We have to show that every $x \in X$ is in the closure of $U := \cap_{n\in\mathbb{N}} U_n$, i.e., for every $x \in X$ and every $\varepsilon > 0$, $B_\varepsilon(x) \cap U \neq \emptyset$. Recall that

$$\overline{B}_\varepsilon(x) := \{y \in X : d(x,y) \leq \varepsilon\},$$
$$B_\varepsilon(x) := \{y \in X : d(x,y) < \varepsilon\},$$

and for every $\varepsilon > 0$, $\overline{B}_\varepsilon(x) \subseteq B_{2\varepsilon}(x)$.

Thus, fix an $x_0 := x \in X$ and an $\varepsilon_0 := \varepsilon > 0$. Since U_1 is dense, $U_1 \cap B_{\varepsilon_0}(x_0) \neq \emptyset$, and therefore we can choose an element x_1 in it. Moreover, since $U_1 \cap B_{\varepsilon_0}(x_0)$ is open, there exists an $\varepsilon_1 > 0$ such that

$$\overline{B}_{\varepsilon_1}(x_1) \subseteq B_{2\varepsilon_1}(x_1) \subseteq U_1 \cap B_{\varepsilon_0}(x_0).$$

Moreover, we can assume without loss of generality that $\varepsilon_1 < 1$. Now repeat this procedure with x_1, ε_1 and U_2, to get an $x_2 \in U_2 \cap B_{\varepsilon_1}(x_1)$ and $0 < \varepsilon_2 < \frac{1}{2}$ such that

$$\overline{B}_{\varepsilon_2}(x_2) \subseteq U_2 \cap B_{\varepsilon_1}(x_1).$$

Continuing this process, we get a sequence of points $x_n \in X$ and positive numbers $0 < \varepsilon_n < \frac{1}{n}$ such that

$$\overline{B}_{\varepsilon_{n+1}}(x_{n+1}) \subseteq U_{n+1} \cap B_{\varepsilon_n}(x_n), \qquad n \in \mathbb{N}.$$

It is easy to see that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Indeed, for given $\delta > 0$, we can choose $N_\delta \in \mathbb{N}$ such that $1/N_\delta < \delta/2$. Then for any $n, m > N_\delta$, since $x_n, x_m \in \overline{B}_{\varepsilon_{N_\delta}}(x_{N_\delta})$, we have $d(x_n, x_m) \leq d(x_n, x_{N_\delta}) + d(x_{N_\delta}, x_m) \leq 2\varepsilon_{N_\delta} < 2/N_\delta < \delta$. Now we use that the space is complete, and hence $(x_n)_{n\in\mathbb{N}}$ converges to a point $x_\infty \in X$. By construction, $x_\infty \in \overline{B}_{\varepsilon_1}(x_1) \subseteq B_\varepsilon(x)$, and also $x_\infty \in \cap_{n\in\mathbb{N}} U_n$, completing the proof. □

A useful consequence of the above theorem is in terms of nowhere dense sets:

3.2 Definition. We say that a set M in a topological space (X, τ) is *nowhere dense* if the closure of M has empty interior.

3.3 Remark. Note that a set M is nowhere dense if and only if its closure is.

3.4 Corollary. A complete metric space is never the countable union of nowhere dense sets.

Proof. Let (X, d) be a complete metric space, and $X = \cup_{n\in\mathbb{N}} C_n$, where each C_n is closed, i.e., $U_n := X \setminus C_n$ is open. Then

$$\cap_{n\in\mathbb{N}} U_n = \cap_{n\in\mathbb{N}} (X \setminus C_n) = X \setminus \cup_{n\in\mathbb{N}} C_n = \emptyset.$$

Hence, by Baire's category theorem, there exists an n such that U_n is not dense, and hence $C_n^\circ = X \setminus \overline{U}_n \neq \emptyset$, i.e., C_n has a non-empty interior. □