ULTRAVIOLET PROPERTIES OF SPINLESS, ONE-PARTICLE YUKAWA MODEL

Dirk - André Deckert

RG Interaction beyween Light & Matter
Mathematisches Institut der LMU

May 27, 2016

Joint work with Alessandro Pizzo (U Roma II)

UV Divergence in Classical Electrodynamics

A classical charge interacting with its field could be modeled by the Maxwell equations:

$$\Box A^\mu(x) = -4\pi e \int_{-\infty}^{\infty} \dot{z}^\mu(\tau)\delta^4(x - z(\tau))\, d\tau, \qquad \partial_\mu A^\mu = 0 \tag{M}$$

coupled to the Lorentz equation:

$$m\ddot{z}^\mu(t) = eF^{\mu\nu}(z(\tau))\dot{z}_\nu(\tau), \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{L}$$

The right-hand side of (L) is ill-defined as at the charge world lines every solution of (M) carries a singularity of the type

$$1/\text{distance}^2.$$

Hence, the coupled set of equations (M) and (L) are ill-defined.

Mathematical Remedy: Unphysical cut off at high frequencies $\Lambda < \infty$ in the fields.

Dirac's Mass Renormalization Method

Dirac's key idea: (M) correct but (L) must be changed.

$$m\ddot{z}^\mu = e\left[F^{\mu\nu}_{\text{free}}\dot{z}_\nu + \underbrace{\frac{1}{2}(F^\Lambda_{\text{ret}} + F^\Lambda_{\text{adv}})^{\mu\nu}\dot{z}_\nu}_{\sim -\frac{1}{2}e\Lambda\ddot{z}^\mu} + \underbrace{\frac{1}{2}(F^\Lambda_{\text{ret}} - F^\Lambda_{\text{adv}})^{\mu\nu}\dot{z}_\nu}_{\sim\frac{2}{3}e(\dddot{z}^\mu\dot{z}^\nu - \dddot{z}^\nu\dot{z}^\mu)\dot{z}_\nu + O(\Lambda^{-1})}\right]$$

For $\Lambda \to \infty$ the divergent term is absorbed by a bare mass $m = m(\Lambda)$ that must tend to $-\infty$ such that

$$m_{\text{ren}} = m(\Lambda) + \frac{1}{2}e\Lambda = \text{experimentally measured mass of an electron}$$

so that effectively we have

$$m_{\text{ren}}\ddot{z}^\mu = eF^{\mu\nu}_{\text{free}}\dot{z}_\nu + \frac{2}{3}e^2(\dddot{z}^\mu\dot{z}^\nu - \dddot{z}^\nu\dot{z}^\mu)\dot{z}_\nu. \qquad \text{(LD)}$$

However, the story does not end here as almost all solutions to (LD) are unphysical and 'good' solutions have to be distinguished from 'bad' ones.

Bosonic UV Divergence in QFT

For some time it had been fashionable to say that because of the failure of the classical theory QFT was invented:

- Hope: Spreading of the wave function provides a natural smearing of the point-like interaction.
- Non-relativistic QED models, there is hope indeed:
  - For the *Nelson model* an energy renormalization is sufficient [Nelson, 1964];
  - For the Pauli-Fierz model it is conjectured [Hiroshima & Spohn 2003]:

$$\frac{m_{\mathrm{eff}}}{m} = \mathcal{O}\left(\left(\frac{\Lambda}{m}\right)^{\gamma}\right)$$

    If so,

$$m(\Lambda) = \mathcal{O}_{\Lambda\to\infty}(\Lambda^{1-\gamma}) \Rightarrow m_{\mathrm{eff}} = \mathrm{const.}$$

- (Pseudo-) Relativistic QED models:
  - No similar results;
  - Not much non-perturbativ information known at all.

Two Results for the Spinless, One-Particle Yukawa Model

We consider the Yukawa model for one nucleon that interacts with its own scalar field. W.r.t. QED we neglect:

- Pair-Creation;
- Spin.

The equation of motion is given by

$$i\frac{d}{dt}\Psi_t = H\Psi_t$$

for the Hamiltonian

$$H := \sqrt{p^2+m^2} + \int \omega(k)a^*(k)a(k)dk + g\int \rho(k)\left(a(k)e^{ikx} + a^*(k)e^{-ikx}\right)\, dk$$

where $\omega(k) := \sqrt{k^2+\mu^2}$ and $\rho(k) := \frac{1}{\sqrt{2\omega(k)}}$.

Introduction of cut-offs

$$H := \sqrt{p^2 + m^2} + \int \omega(k) a^*(k) a(k)\, dk + g \int \rho(k) \left(a(k) e^{ikx} + a^*(k) e^{-ikx} \right)\, dk$$

As in classical electrodynamics the equation of motion is ill-defined because

$$\rho \notin L^2.$$

To study this ill-defined equation of motion we introduce:

- A smearing of the point interaction on small lengths by cutting off the interaction at hight momenta Λ;
- A cut-off of the interaction for momenta below $\kappa = 1$ to separate the ultraviolet from the infrared problem – the latter of which is well understood.

In the P-fibre of the total momentum operator $P = p + P^f$ for

$$P^f = \int k\, a^*(k) a(k) dk$$

the model Hamiltonian reads

$$H_P|_\kappa^\Lambda = \underbrace{\sqrt{(P - P^f)^2 + m^2}}_{=: H^{nuc}} + \underbrace{\int \omega(k) b^*(k) b(k) dk}_{=: H^f}$$

$$+ g \underbrace{\int_{\kappa \leq |k| \leq \Lambda} \rho(k) \left(b(k) + b^*(k) \right) dk}_{=: \Phi|_\kappa^\Lambda = \phi|_\kappa^\Lambda + \phi^*|_\kappa^\Lambda}$$

on the boson Fock space

$$\mathcal{F}|_\kappa^\Lambda := \bigoplus_{j=0}^{\infty} \mathcal{F}^{(j)}, \qquad \mathcal{F}^{(0)} := \mathbb{C}, \qquad \mathcal{F}^{j \geq 1} := \bigodot_{l=1}^{j} L^2(\mathcal{B}_\Lambda \setminus \mathcal{B}_\kappa, \mathbb{C}; dk).$$

Ultraviolet Behavior of the Energy

For the ground state energy in the P fibre

$$E_P|_\kappa^\Lambda := \inf \sigma \left(H_P|_\kappa^\Lambda \upharpoonright \mathcal{F}|_\kappa^\Lambda \right)$$

we find

Theorem (D., Pizzo; CMP 2014)

Let $|P| \leq P_{max} < \infty$ and $|g|$ sufficiently small. There are constants $0 \leq b \leq a < \infty$ such that for all $\kappa \leq \Lambda < \infty$

$$-g^2 a\Lambda \leq E_{P,\Lambda} - \sqrt{P^2 + m^2} \leq -g^2 b\Lambda.$$

- Despite the quantum dispersion relation the energy diverges linearly as in the classical analogue;
- In [Lieb & Loss, 2000] also a linear dependence of the self-energy was shown for

$$H = \sqrt{(p - \sqrt{\alpha}A(x))^2 + m^2}.$$

Ultraviolet Behavior of the Effective Velocity

Theorem (D., Pizzo; CMP 2014)

Let $|P| \leq P_{max} < \infty$ and $|g|$ be sufficiently small. Then, there exist universal constants $C_1, C_2 > 0$ such that the following estimate holds true for all $\kappa \leq \Lambda < \infty$:

$$\left| \frac{\partial E_P|_\kappa^\Lambda}{\partial P_i} \right| \leq \Lambda^{-g^2 C_1} \frac{|P|}{[P^2 + m^2]^{1/2}} + C_2 |g|^{1/2}, \qquad i = 1, 2, 3.$$

Consequences:

- As the free energy is given by $E_P|_\Lambda^\Lambda = \frac{|P|}{[P^2+m^2]}$, switching on the interaction for arbitrary small $|g| > 0$ flattens the mass shell up to $O(|g|^{1/2})$;
- No matter how small the coupling constant is, the nucleon becomes infinitely heavy in the limit $\Lambda \to \infty$ and the theory becomes trivial;
- No choice of mass renormalization $m = m(\Lambda)$ can prevent this behavior!

Strategy of Proof

Problem

Regular perturbation theory would require $g = O(\Lambda^{-1})$,
but we want results uniform in $\kappa \leq \Lambda < \infty$.

We slice up the domain of the interaction integral

$$g\Phi|_{\kappa}^{\Lambda} = \sum_{n=1}^{N} g \int_{\Lambda\gamma^n \leq |k| \leq \Lambda\gamma^{n-1}} \rho(k)\,(b(k) + b^*(k))\;dk = g\sum_{n=1}^{N} \Phi|_{\Lambda\gamma^{n-1}}^{\Lambda\gamma^n}$$

with respect to a fineness parameter $\frac{1}{2} < \gamma < 1$, chosen such that

$$1 = \kappa = \Lambda\gamma^N \qquad \Rightarrow \qquad N = \frac{\ln \Lambda}{-\ln\gamma} \sim \ln\Lambda(1-\gamma)$$

and define

$$H_{P,n} := H^{nuc} + H^f + \Phi|_{\Lambda\gamma^n}^{\Lambda}, \qquad \mathcal{F}_n := \mathcal{F}|_{\Lambda\gamma^n}^{\Lambda}.$$

Iterative Construction of the P-fibre Ground State

The construction of the ground state $\Psi_{P,N}$ of $H_{P,N} \upharpoonright \mathcal{F}_N$ is done by induction adding slices of the interaction step-by-step starting from the free ground state $\Psi_{P,0}$ of $H_{P,0} \upharpoonright \mathcal{F}_0$.

Assume at step $(n-1)$:

(i) $\Psi_{P,n-1}$ and $E_{P,n-1}$ are unique ground state and energy of $H_{P,n-1} \upharpoonright \mathcal{F}_{n-1}$;

(ii) For a universal constant $\zeta > 0$ the spectral gap fulfills

$$\mathrm{Gap}\left(H_{P,n-1} \upharpoonright \mathcal{F}_{n-1}\right) \geq \zeta\omega\left(\Lambda\gamma^n\right).$$

Induction step to n:

(1) External information needed: $\mathrm{Gap}\left(H_{P,n-1} \upharpoonright \mathcal{F}_n\right) \geq \zeta\omega\left(\Lambda\gamma^n\right)$ and $E_{P,n} \leq E_{P,n-1}$;

(2) Neumann expansion of ground state $\Psi_{P,n}$ with respect to ground state $\Psi_{P,n-1}$ and interaction slice $\Phi|_n^{n-1}$.

Neumann expansion of the Ground State

Intended expansion:

$$\Psi_{P,n} := -\frac{1}{2\pi i}\oint_{\Gamma_n} \frac{dz}{H_{P,n}-z}\Psi_{P,n-1}$$

$$= -\frac{1}{2\pi i}\sum_{j=0}^{\infty}\oint_{\Gamma_n} \frac{dz}{H_{P,n-1}-z}\left[-g\Phi|_{n-1}^{n}\frac{1}{H_{P,n-1}-z}\right]^j \Psi_{P,n-1}$$

For this we need an estimate of:

$$\left\| \left(\frac{1}{H_{P,n-1}-z}\right)^{1/2} g\Phi|_{n}^{n-1}\left(\frac{1}{H_{P,n-1}-z}\right)^{1/2}\right\|_{\mathcal{F}_n} = \mathcal{O}(|g|)$$

for a convenient contour $z \in \Gamma_n$ and uniformly in $\kappa \leq \Lambda < \infty$.

1. By a variational argument ensures $\mathrm{Gap}\left(H_{P,n-1}\restriction\mathcal{F}_n\right)\geq\zeta\omega\left(\Lambda\gamma^n\right)$.
2. Let us restrict to contours $z\in\Gamma_n$ in $\mathbb{C}$ such that

$$\frac{1}{2}\zeta\omega\left(\Lambda\gamma^{n+1}\right)\leq|E_{P,n-1}-z|\leq\zeta\omega\left(\Lambda\gamma^{n+1}\right).$$

3. The iteration only works well when adding the interaction slices starting from $\Lambda\gamma^0$ to $\Lambda\gamma^N=1$ in decreasing order as then

$$\left\|g\phi|_n^{n-1}\left(\frac{1}{H_{P,n-1}-z}\right)^{1/2}\right\|_{\mathcal{F}_n}=\mathcal{O}\left(|g|\left(\Lambda\gamma^{n-1}(1-\gamma)\right)^{1/2}\right),$$

is compensated thanks to the spectral gap estimate and the chosen domain for z which gives

$$\left\|\left(\frac{1}{H_{P,n-1}-z}\right)^{1/2}\right\|_{\mathcal{F}_n}=\mathcal{O}\left(\left(\frac{1}{\Lambda\gamma^{n+1}(1-\gamma)}\right)^{1/2}\right).$$

4. This allows the construction of $\Psi_{P,n}$ and another variational argument guarantees $E_{P,n}\leq E_{P,n-1}$ so that by Kato's theorem

$$\mathrm{Gap}\left(H_{P,n}\restriction\mathcal{F}_n\right)\geq\zeta\omega\left(\Lambda\gamma^{n+1}\right)$$

which closes the induction.

Recursion Formula for the Expansion

The ground state of $H_{P,n} \upharpoonright \mathcal{F}_n$ for sufficiently small $|g|$ is then given by

$$\Psi_{P,n} := -\frac{1}{2\pi i}\oint_{\Gamma_n} \frac{dz}{H_{P,n} - z}\Psi_{P,n-1}$$
$$= -\frac{1}{2\pi i}\sum_{j=0}^{\infty}\oint_{\Gamma_n} \frac{dz}{H_{P,n-1} - z}\left[-g\Phi|_{n-1}^{n}\frac{1}{H_{P,n-1} - z}\right]^j \Psi_{P,n-1}$$

where

$$\Gamma_n := \left\{ z \in \mathbb{C} \,\middle|\, |E_{P,n-1} - z| = \frac{1}{2}\zeta\omega\left(\Lambda\gamma^{n+1}\right)\right\}.$$

This provides the key estimate for $z \in \Gamma_n$ for the error control:

$$\left\| \left(\frac{1}{H_{P,n-1} - z}\right)^{1/2} g\Phi|_{n}^{n-1}\left(\frac{1}{H_{P,n-1} - z}\right)^{1/2}\right\|_{\mathcal{F}_n} \leq \mathcal{O}\left(|g|(1-\gamma)^{1/2}\right).$$

The two main results must now also be inferred by iterative expansion.

The $\mathcal{F}_n$-distance between ground state vectors in the sequence $(\Psi_{P,n})_{n\in\mathbb{N}}$ can now be controlled explicitly by the von Neumann expansion (here, up to third order):

$$\begin{aligned}\Psi_{P,n} =&\Psi_{P,n-1} - g\frac{1}{H_{P,n-1}-E_{P,n-1}}\phi^*|_n^{n-1}\Psi_{P,n-1}\\ &+ g^2\widetilde{\mathcal{Q}}^{\perp}_{P,n-1}\frac{1}{H_{P,n-1}-E_{P,n-1}}\phi|_n^{n-1}\frac{1}{H_{P,n-1}-E_{P,n-1}}\phi^*|_n^{n-1}\Psi_{P,n-1}\\ &+ g^2\widetilde{\mathcal{Q}}^{\perp}_{P,n-1}\frac{1}{H_{P,n-1}-E_{P,n-1}}\phi^*|_n^{n-1}\frac{1}{H_{P,n-1}-E_{P,n-1}}\phi^*|_n^{n-1}\Psi_{P,n-1}\\ &- g^2\widetilde{\mathcal{Q}}_{P,n-1}\phi|_n^{n-1}\left(\frac{1}{H_{P,n-1}-E_{P,n-1}}\right)^2\phi^*|_n^{n-1}\Psi_{P,n-1}\\ &+\mathcal{O}\left(|g|^3(1-\gamma)^{3/2}\right),\end{aligned}$$

where $\widetilde{\mathcal{Q}}_{P,n-1}$ is the orthogonal projector on $\Psi_{P,n-1}\otimes\Omega\in\mathcal{F}_n$.

Expansion of the Effective Velocity

Using this formula one may start expanding the effective velocity at level n

$$\frac{\partial E_{P,n}}{\partial P_i} = \left\langle \widehat{\Psi}_{P,n}, V_i(P)\widehat{\Psi}_{P,n} \right\rangle , \qquad V_i(P) := \frac{P_i - P_i^f}{\left[(P - P^f)^2 + m^2\right]^{1/2}}$$

by expanding the vectors $\Psi_{P,n}$ in terms of $\Psi_{P,n-1}$.

The aim is to find a flow equation relating the effective velocity of level n to the one of level $(n-1)$.

$$\left\langle \widehat{\Psi}_{P,n}, V_i(P)\widehat{\Psi}_{P,n} \right\rangle =$$

$$\left(1 - g^2 \alpha_P|_n^{n-1} + \mathcal{O}\left(\left[|g|(1-\gamma)^{1/2}\right]^4\right)\right) \left\langle \widehat{\Psi}_{P,n-1}, V_i(P)\widehat{\Psi}_{P,n-1} \right\rangle$$

$$+ A_{P,n-1} + B_{P,n-1} + \mathcal{O}\left(\left[|g|(1-\gamma)^{1/2}\right]^4\right),$$

where

$$A_{P,n-1} :=$$

$$g^2 \left\langle \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi^*|_n^{n-1} \widehat{\Psi}_{P,n-1}, V_i(P) \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi^*|_n^{n-1} \widehat{\Psi}_{P,n-1} \right\rangle$$

$$B_{P,n-1} :=$$

$$g^2 2\mathfrak{Re}\left\langle \mathcal{Q}^{\perp}_{P,n-1} \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi|_n^{n-1} \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi^*|_n^{n-1} \widehat{\Psi}_{P,n-1}, \times \right.$$

$$\left. \times \; V_i(P)\widehat{\Psi}_{P,n-1} \right\rangle$$

$$\left\langle \widehat{\Psi}_{P,N}, V_i(P)\widehat{\Psi}_{P,N} \right\rangle = \overbrace{\prod_{j=1}^{N}\left(1 - g^2\alpha_P\big|_{N-j+1}^{N-j}\right)}^{\le e^{-c_1 g^2 N(1-\gamma)} \le \Lambda^{-g^2 c_1}} \overbrace{\left\langle \widehat{\Psi}_{P,0}, V_i(P)\widehat{\Psi}_{P,0} \right\rangle}^{=\frac{|P|}{\sqrt{|P|^2+m^2}}}$$

$$+ \sum_{j=2}^{N-1} \left(1 - g^2\alpha_P\big|_N^{N-1}\right)\dots\left(1 - g^2\alpha_P\big|_{N-j+1}^{N-j}\right)\left[A_{P,N-j-1} + B_{P,N-j-1}\right]$$

$$+ \left(1 - g^2\alpha_P\big|_N^{N-1}\right)\left[A_{P,N-2} + B_{P,N-2}\right] + \left[A_{P,N-1} + B_{P,N-1}\right]$$

$$+ \mathcal{O}\left(\underbrace{N}_{\le \frac{\ln\Lambda}{1-\gamma}}\left[|g|(1-\gamma)^{1/2}\right]^4\right).$$

Crucial variational estimate:

$$c_1 g^2(1-\gamma) \le g^2\alpha_P\big|_n^{n-1} \le c_2 g^2(1-\gamma).$$

If one could also provide the bounds

$$|A_{P,N-j}| \le g^2 C\frac{1-\gamma}{\Lambda\gamma^{N-j+1}}, \qquad |B_{P,N-j}| \le |g|^{5/2}C(1-\gamma),$$

$$\left\langle \widehat{\Psi}_{P,N}, V_i(P)\widehat{\Psi}_{P,N} \right\rangle = \overbrace{\prod_{j=1}^{N} \left(1 - g^2\alpha_P\big|_{N-j+1}^{N-j}\right)}^{\leq e^{-c_1 g^2 N(1-\gamma)} \leq \Lambda^{-g^2 c_1}} \overbrace{\left\langle \widehat{\Psi}_{P,0}, V_i(P)\widehat{\Psi}_{P,0} \right\rangle}^{= \frac{|P|}{\sqrt{|P|^2+m^2}}}$$

$$+ \sum_{j=2}^{N-1} \left(1 - g^2\alpha_P\big|_{N}^{N-1}\right) \ldots \left(1 - g^2\alpha_P\big|_{N-j+1}^{N-j}\right) \left[A_{P,N-j-1} + B_{P,N-j-1}\right]$$

$$+ \left(1 - g^2\alpha_P\big|_{N}^{N-1}\right) \left[A_{P,N-2} + B_{P,N-2}\right] + \left[A_{P,N-1} + B_{P,N-1}\right]$$

$$+ \mathcal{O}\left(\underbrace{N}_{\leq \frac{\ln \Lambda}{1-\gamma}} \left[|g|(1-\gamma)^{1/2}\right]^4 \right).$$

Crucial variational estimate:

$$c_1 g^2(1-\gamma) \leq g^2\alpha_P\big|_{n}^{n-1} \leq c_2 g^2(1-\gamma).$$

If one could also provide the bounds

$$|A_{P,N-j}| \leq g^2 C \frac{1-\gamma}{\Lambda\gamma^{N-j+1}}, \qquad |B_{P,N-j}| \leq |g|^{5/2} C(1-\gamma),$$

$$\left\langle \widehat{\Psi}_{P,N}, V_i(P)\widehat{\Psi}_{P,N}\right\rangle = \overbrace{\prod_{j=1}^{N}\left(1 - g^2\alpha_P|_{N-j+1}^{N-j}\right)}^{\leq e^{-c_1 g^2 N(1-\gamma)} \leq \Lambda^{-g^2 c_1}} \overbrace{\left\langle \widehat{\Psi}_{P,0}, V_i(P)\widehat{\Psi}_{P,0}\right\rangle}^{=\frac{|P|}{\sqrt{|P|^2+m^2}}}$$

$$+ \sum_{j=2}^{N-1}\left(1 - g^2\alpha_P|_{N}^{N-1}\right)\ldots\left(1 - g^2\alpha_P|_{N-j+1}^{N-j}\right)\left[A_{P,N-j-1} + B_{P,N-j-1}\right]$$

$$+ \left(1 - g^2\alpha_P|_{N}^{N-1}\right)\left[A_{P,N-2} + B_{P,N-2}\right] + \left[A_{P,N-1} + B_{P,N-1}\right]$$

$$+ \mathcal{O}\left(\underbrace{N}_{\leq \frac{\ln\Lambda}{1-\gamma}}\left[|g|(1-\gamma)^{1/2}\right]^4\right).$$

Crucial variational estimate:

$$c_1 g^2(1-\gamma) \leq g^2\alpha_P|_{n}^{n-1} \leq c_2 g^2(1-\gamma).$$

If one could also provide the bounds

$$|A_{P,N-j}| \leq g^2 C\frac{1-\gamma}{\Lambda\gamma^{N-j+1}}, \qquad |B_{P,N-j}| \leq |g|^{5/2}C(1-\gamma),$$

$$\left\langle \widehat{\Psi}_{P,N}, V_i(P)\widehat{\Psi}_{P,N} \right\rangle = \overbrace{\prod_{j=1}^{N}\left(1 - g^2\alpha_P|_{N-j+1}^{N-j}\right)}^{\leq e^{-c_1 g^2 N(1-\gamma)} \leq \Lambda^{-g^2 c_1}} \overbrace{\left\langle \widehat{\Psi}_{P,0}, V_i(P)\widehat{\Psi}_{P,0} \right\rangle}^{=\frac{|P|}{\sqrt{|P|^2+m^2}}}$$

$$+ \sum_{j=2}^{N-1} \left(1 - g^2\alpha_P|_N^{N-1}\right) \dots \left(1 - g^2\alpha_P|_{N-j+1}^{N-j}\right) \left[A_{P,N-j-1} + B_{P,N-j-1}\right]$$

$$+ \left(1 - g^2\alpha_P|_N^{N-1}\right)\left[A_{P,N-2} + B_{P,N-2}\right] + \left[A_{P,N-1} + B_{P,N-1}\right]$$

$$+ \mathcal{O}\left(\underbrace{N}_{\leq \frac{\ln\Lambda}{1-\gamma}} \left[|g|(1-\gamma)^{1/2}\right]^4 \right).$$

Crucial variational estimate:

$$c_1 g^2(1-\gamma) \leq g^2\alpha_P|_n^{n-1} \leq c_2 g^2(1-\gamma).$$

If one could also provide the bounds

$$|A_{P,N-j}| \leq g^2 C \frac{1-\gamma}{\Lambda\gamma^{N-j+1}}, \qquad |B_{P,N-j}| \leq |g|^{5/2} C(1-\gamma),$$

this would imply

$$\left|\frac{\partial E_P|_\kappa^\Lambda}{\partial P_i}\right| \leq \Lambda^{-g^2 C_1} \frac{|P|}{[P^2+m^2]^{1/2}} + C_2|g|^{1/2} + \mathcal{O}\left(|g|^4 \log\Lambda(1-\gamma)\right).$$

The real hard part is to show these bounds

$$|A_{P,N-j}| \leq g^2 C \frac{1-\gamma}{\Lambda\gamma^{N-j+1}}, \qquad |B_{P,N-j}| \leq |g|^{5/2} C(1-\gamma).$$

$$A_{P,n-1} := g^2 \left\langle \frac{1}{H_{P,n-1}-E_{P,n-1}} \phi^*|_n^{n-1} \widehat{\Psi}_{P,n-1}, V_i(P) \frac{1}{H_{P,n-1}-E_{P,n-1}} \phi^*|_n^{n-1} \widehat{\Psi}_{P,n-1} \right\rangle$$

$$B_{P,n-1} := g^2 2\mathfrak{Re}\left\langle \mathcal{Q}^\perp_{P,n-1} \frac{1}{H_{P,n-1}-E_{P,n-1}} \phi|_n^{n-1} \frac{1}{H_{P,n-1}-E_{P,n-1}} \phi^*|_n^{n-1} \widehat{\Psi}_{P,n-1}, \times \times V_i(P) \widehat{\Psi}_{P,n-1} \right\rangle$$

$|B_{P,N-j}| \leq |g|^{5/2} C(1-\gamma)$

$$B_{P,n-1} := g^2 2\mathfrak{Re}\left\langle \mathcal{Q}^{\perp}_{P,n-1} \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi|_n^{n-1} \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi^*|_n^{n-1} \widehat{\Psi}_{P,n-1}, \times \times V_i(P)\widehat{\Psi}_{P,n-1} \right\rangle$$

The situation is worse because we cannot use V.

$$|B_{P,n-1}| = g^2 \left| 2\mathfrak{Re} \int_{\Lambda\gamma^n}^{\Lambda\gamma^{n-1}} dk\, \rho(k)^2 \times \times \left\langle \mathcal{Q}^{\perp}_{P,n-1} \frac{1}{H_{P,n-1} - E_{P,n-1}} \frac{1}{H_{P-k,n-1} + \omega(k) - E_{P,n-1}} \widehat{\Psi}_{P,n-1}, V_i(P)\widehat{\Psi}_{P,n-1} \right\rangle \right|$$

$$\leq g^2 C \int_{\Lambda\gamma^n}^{\Lambda\gamma^{n-1}} dk\, \frac{1}{k^2} \left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \mathcal{Q}^{\perp}_{P,n-1} V_i(P)\widehat{\Psi}_{P,n-1} \right\|$$

$$\leq g^2 C \Lambda\gamma^{n-1}(1-\gamma) \left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \mathcal{Q}^{\perp}_{P,n-1} V_i(P)\widehat{\Psi}_{P,n-1} \right\|.$$

$|B_{P,N-j}| \leq |g|^{5/2} C(1-\gamma)$

$$B_{P,n-1} := \\ g^2 2\mathfrak{Re}\left\langle \mathcal{Q}^{\perp}_{P,n-1} \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi|_n^{n-1} \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi^*|_n^{n-1} \widehat{\Psi}_{P,n-1}, \times \\ \times V_i(P)\widehat{\Psi}_{P,n-1} \right\rangle$$

The situation is worse because we cannot use V.

$$|B_{P,n-1}| = g^2 \left| 2\mathfrak{Re} \int_{\Lambda\gamma^n}^{\Lambda\gamma^{n-1}} dk\, \rho(k)^2 \times \right.$$

$$\left. \times \left\langle \mathcal{Q}^{\perp}_{P,n-1} \frac{1}{H_{P,n-1} - E_{P,n-1}} \frac{1}{H_{P-k,n-1} + \omega(k) - E_{P,n-1}} \widehat{\Psi}_{P,n-1}, V_i(P)\widehat{\Psi}_{P,n-1} \right\rangle \right|$$

$$\leq g^2 C \int_{\Lambda\gamma^n}^{\Lambda\gamma^{n-1}} dk\, \frac{1}{k^2} \left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \mathcal{Q}^{\perp}_{P,n-1} V_i(P)\widehat{\Psi}_{P,n-1} \right\|$$

$$\leq g^2 C \Lambda \gamma^{n-1}(1-\gamma) \left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \mathcal{Q}^{\perp}_{P,n-1} V_i(P)\widehat{\Psi}_{P,n-1} \right\|.$$

We need to show

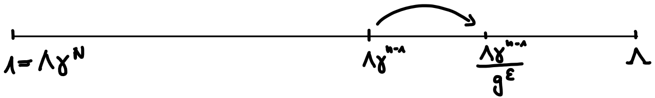
$$\left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \mathcal{Q}^{\perp}_{P,n-1} V_i(P) \widehat{\Psi}_{P,n-1} \right\| \leq C \frac{|g|^{1/2}}{\Lambda \gamma^n}.$$

To show this we use another expansion from scale $\Lambda\gamma^{n-1}$ to scale

$$\Xi_{n-1} := \Lambda \gamma^l$$

for an $l \in \mathbb{N} \cup \{0\}$ such that

$$\Lambda\gamma^l \leq \min\left\{\Lambda, \frac{\Lambda\gamma^{n-1}}{g^{\epsilon}}\right\} < \Lambda\gamma^{l-1}.$$



Backwards Expansion

Using the control of the mass shell from the construction one infers

$$\left\| \left(\frac{1}{H_{P,\Xi_{n-1}} - z} \right)^{1/2} g\Phi|_{\Lambda\gamma^{n-1}}^{\Xi_{n-1}} \left(\frac{1}{H_{P,\Xi_{n-1}} - z} \right)^{1/2} \right\|_{\mathcal{F}_{n-1}} \leq |g|^{1-\frac{\epsilon}{2}} C, \quad z \in \Gamma_{P,n-1}.$$

and after expansion

$$\left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \mathcal{Q}_{P,n-1}^{\perp} V_i(P) \widehat{\Psi}_{P,n-1} \right\| \leq \left\| \mathcal{Q}_{P,\Xi_{n-1}}^{\perp} \frac{1}{H_{P,\Xi_{n-1}} - E_{P,n-1}} V_i(P) \widehat{\Psi}_{P,\Xi_{n-1}} \right\| + C \frac{|g|^{1-\frac{\epsilon}{2}}}{\Lambda\gamma^n}.$$

1. Case $\Xi_{n-1} < \Lambda$. In this case we exploit

$$\left\| \mathcal{Q}^{\perp}_{P,\Xi_{n-1}} \frac{1}{H_{P,\Xi_{n-1}} - E_{P,n-1}} \right\|_{\mathcal{F}|^{\Lambda}_{\Xi_{n-1}}} \leq C \frac{g^{\epsilon}}{\Lambda \gamma^{n}}$$

2. Case $\Xi_{n-1} = \Lambda$. In this case we have

$$\mathcal{Q}^{\perp}_{P,\Xi_{n-1}} V_i(P) \widehat{\Psi}_{P,\Xi_{n-1}} = \frac{P_i}{\sqrt{P^2 + m^2}} \mathcal{Q}^{\perp}_{P,\Xi_{n-1}} \widehat{\Psi}_{P,\Xi_{n-1}} = 0.$$

   Therefore

$$\left\| \mathcal{Q}^{\perp}_{P,\Xi_{n-1}} \frac{1}{H_{P,\Xi_{n-1}} - E_{P,n-1}} \mathcal{Q}^{\perp}_{P,\Xi_{n-1}} V_i(P) \widehat{\Psi}_{P,\Xi_{n-1}} \right\| \leq C \frac{g^{\epsilon}}{\Lambda \gamma^{n}}.$$

Therefore,

$$|B_{P,n-1}| \leq g^2 C \Lambda \gamma^{n-1} (1-\gamma) \left(C \frac{g^{\epsilon}}{\Lambda \gamma^{n}} + C \frac{|g|^{1-\frac{\epsilon}{2}}}{\Lambda \gamma^{n}} \right) \leq g^{5/2} C (1-\gamma), \quad \text{for } \epsilon = \frac{1}{2}.$$

Outlook

- In which scaling can we prevent the model from becoming trivial?
- How to remove the technical artifact of the $|g|^{1/2}$ error term?
- How to infer bounds from below?

Thank you!

Expansion of the Self-Energy

In this way we can readily control the shift in the ground state energy:

$$E_{P,n} - E_{P,n-1} = \frac{\langle \Psi_{P,n}, [H_{P,n} - H_{P,n-1}]\, \Psi_{P,n-1} \rangle}{\langle \Psi_{P,n}, \Psi_{P,n-1} \rangle}$$

$$= \frac{\left\langle \Psi_{P,n}, g\Phi|_n^{n-1} \Psi_{P,n-1} \right\rangle}{\langle \Psi_{P,n}, \Psi_{P,n-1} \rangle}$$

$$= \Delta E_P|_n^{n-1} + \mathcal{O}\left(|g|^4 \Lambda (1-\gamma)^{4/2} \right).$$

for

$$\Delta E_P|_n^{n-1} = g^2 \int_{\Lambda\gamma^n}^{\Lambda\gamma^{n-1}} dk\, \rho(k)^2 \left\langle \widehat{\Psi}_{P,n-1}, \frac{1}{H_{P-k,n-1} + \omega(k) - E_{P,n-1}} \widehat{\Psi}_{P,n-1} \right\rangle.$$

Crucial variational estimate:

$$a\, g^2 \Lambda \gamma^{n-1}(1-\gamma) \leq \Delta E_P|_n^{n-1} \leq b\, g^2 \Lambda \gamma^{n-1}(1-\gamma).$$

Expansion of the Self-Energy

In this way we can readily control the shift in the ground state energy:

$$E_{P,n} - E_{P,n-1} = \frac{\langle \Psi_{P,n}, [H_{P,n} - H_{P,n-1}]\, \Psi_{P,n-1} \rangle}{\langle \Psi_{P,n}, \Psi_{P,n-1} \rangle}$$

$$= \frac{\left\langle \Psi_{P,n}, g\Phi|_n^{n-1} \Psi_{P,n-1} \right\rangle}{\langle \Psi_{P,n}, \Psi_{P,n-1} \rangle}$$

$$= \Delta E_P|_n^{n-1} + \mathcal{O}\left(|g|^4 \Lambda (1-\gamma)^{4/2} \right).$$

for

$$\Delta E_P|_n^{n-1} = g^2 \int_{\Lambda\gamma^n}^{\Lambda\gamma^{n-1}} dk\, \rho(k)^2 \left\langle \widehat{\Psi}_{P,n-1}, \frac{1}{H_{P-k,n-1} + \omega(k) - E_{P,n-1}} \widehat{\Psi}_{P,n-1} \right\rangle.$$

Crucial variational estimate:

$$a\, g^2 \Lambda \gamma^{n-1}(1-\gamma) \leq \Delta E_P|_n^{n-1} \leq b\, g^2 \Lambda \gamma^{n-1}(1-\gamma).$$

Expansion of the Self-Energy

In this way we can readily control the shift in the ground state energy:

$$
\begin{aligned}
E_{P,n} - E_{P,n-1} &= \frac{\langle \Psi_{P,n}, [H_{P,n} - H_{P,n-1}]\, \Psi_{P,n-1} \rangle}{\langle \Psi_{P,n}, \Psi_{P,n-1} \rangle} \\
&= \frac{\left\langle \Psi_{P,n}, g\Phi|_n^{n-1} \Psi_{P,n-1} \right\rangle}{\langle \Psi_{P,n}, \Psi_{P,n-1} \rangle} \\
&= \Delta E_P|_n^{n-1} + \mathcal{O}\left(|g|^4 \Lambda (1-\gamma)^{4/2} \right).
\end{aligned}
$$

for

$$
\Delta E_P|_n^{n-1} = g^2 \int_{\Lambda\gamma^n}^{\Lambda\gamma^{n-1}} dk\, \rho(k)^2 \left\langle \widehat{\Psi}_{P,n-1}, \frac{1}{H_{P-k,n-1} + \omega(k) - E_{P,n-1}} \widehat{\Psi}_{P,n-1} \right\rangle .
$$

Crucial variational estimate:

$$
a\, g^2 \Lambda \gamma^{n-1} (1-\gamma) \le \Delta E_P|_n^{n-1} \le b\, g^2 \Lambda \gamma^{n-1} (1-\gamma).
$$

Expansion of the Self-Energy

In this way we can readily control the shift in the ground state energy:

$$
\begin{aligned}
E_{P,n} - E_{P,n-1} &= \frac{\left\langle \Psi_{P,n}, \left[H_{P,n} - H_{P,n-1} \right] \Psi_{P,n-1} \right\rangle}{\left\langle \Psi_{P,n}, \Psi_{P,n-1} \right\rangle} \\
&= \frac{\left\langle \Psi_{P,n}, g\Phi|_n^{n-1} \Psi_{P,n-1} \right\rangle}{\left\langle \Psi_{P,n}, \Psi_{P,n-1} \right\rangle} \\
&= \Delta E_P|_n^{n-1} + \mathcal{O}\left(|g|^4 \Lambda (1-\gamma)^{4/2} \right).
\end{aligned}
$$

for

$$
\Delta E_P|_n^{n-1} = g^2 \int_{\Lambda\gamma^n}^{\Lambda\gamma^{n-1}} dk\, \rho(k)^2 \left\langle \widehat{\Psi}_{P,n-1}, \frac{1}{H_{P-k,n-1} + \omega(k) - E_{P,n-1}} \widehat{\Psi}_{P,n-1} \right\rangle .
$$

Crucial variational estimate:

$$
a\, g^2 \Lambda \gamma^{n-1} (1-\gamma) \leq \Delta E_P|_n^{n-1} \leq b\, g^2 \Lambda \gamma^{n-1} (1-\gamma).
$$

This together with

$$E_{P,N} = E_{P,0} - \sum_{n=1}^{N} \Delta E_P|_n^{n-1} + \mathcal{O}\left(N|g|^4(1-\gamma)^{4/2}\right),$$

and using $N \leq \frac{\ln\Lambda}{(1-\gamma)}$ implies

$$E_{P,N} \leq \sqrt{P^2+m^2} - g^2 a\Lambda(1-\gamma)\sum_{n=1}^{N}\gamma^{n-1} + \mathcal{O}\left(\ln\Lambda|g|^4(1-\gamma)\right)$$

as well as

$$E_{P,N} \geq \sqrt{P^2+m^2} - g^2 b\Lambda(1-\gamma)\sum_{n=1}^{N}\gamma^{n-1} - \mathcal{O}\left(\ln\Lambda|g|^4(1-\gamma)\right)$$

for which the errors can be controlled by $\gamma \to 1$.

This together with

$$E_{P,N} = E_{P,0} - \sum_{n=1}^{N} \Delta E_P|_n^{n-1} + \mathcal{O}\left(N|g|^4(1-\gamma)^{4/2}\right),$$

and using $N \leq \frac{\ln \Lambda}{(1-\gamma)}$ implies

$$E_{P,N} \leq \sqrt{P^2 + m^2} - g^2 a\Lambda(1-\gamma)\sum_{n=1}^{N}\gamma^{n-1} + \mathcal{O}\left(\ln \Lambda |g|^4(1-\gamma)\right)$$

as well as

$$E_{P,N} \geq \sqrt{P^2 + m^2} - g^2 b\Lambda(1-\gamma)\sum_{n=1}^{N}\gamma^{n-1} - \mathcal{O}\left(\ln \Lambda |g|^4(1-\gamma)\right)$$

for which the errors can be controlled by $\gamma \to 1$.

$|A_{P,N-j}| \leq g^2 C \frac{1-\gamma}{\Lambda\gamma^{N-j+1}}$

$$A_{P,n-1} := g^2 \left\langle \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi^* |_n^{n-1} \widehat{\Psi}_{P,n-1}, V_i(P) \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi^* |_n^{n-1} \widehat{\Psi}_{P,n-1} \right\rangle$$

A first estimate after pull-through gives

$$\begin{aligned} |A_{P,n-1}| = \; & g^2 \int_{\Lambda\gamma^{n-1}}^{\Lambda\gamma^n} d^3k \, \rho(k)^2 \Big\langle \frac{1}{H_{P,n-1} + \omega(k) - E_{P,n-1}} \widehat{\Psi}_{P,n-1}, \\ & \times V_i(P-k) \frac{1}{H_{P-k,n-1} + \omega(k) - E_{P,n-1}} \widehat{\Psi}_{P,n-1} \Big\rangle \\ & \leq C g^2 \int_{\Lambda\gamma^{n-1}}^{\Lambda\gamma^n} d|k| k^2 \frac{1}{|k|} \frac{1}{|k|} \frac{1}{|k|} \leq C g^2 (1-\gamma) \end{aligned}$$

$\left|A_{P,N-j}\right| \leq g^2 C \frac{1-\gamma}{\Lambda \gamma^{N-j+1}}$

This is too coarse and one has to do better:

$$A_{P,n-1} = A_{P,n-1}\Big|_{P=0} +$$

$$\int_0^1 d\lambda \frac{d}{d\lambda} g^2 \left\langle \frac{1}{H_{P\lambda,n-1} - E_{P\lambda,n-1}} \phi^* |_n^{n-1} \widehat{\Psi}_{P\lambda,n-1}, \right.$$

$$\left. \times V_i(P\lambda) \frac{1}{H_{P\lambda,n-1} - E_{P\lambda,n-1}} \phi^* |_n^{n-1} \widehat{\Psi}_{P\lambda,n-1} \right\rangle$$

- $A_{P,n-1}\Big|_{P=0} = 0$ due to rotational symmetry.
- For each derivatives of the resolvents or the ground state vector we gain another resolvent.
- The derivative of V gives

$$\frac{d}{d\lambda} V_i(P\lambda) = \frac{P_i\lambda - V_i(P\lambda) \sum_{j=1}^{3} V_j(P\lambda) P_j \lambda}{\sqrt{(P\lambda - P^f)^2 + m^2}}.$$