Braided categories of endomorphisms in QFT[1]

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München, 27-28 May 2016, 38th LQP Workshop

[1]joint work with K.-H. Rehren, see [arXiv:1512.01995v1]

after [Doplicher, Haag, Roberts 69-74]:

RCFTs (as Haag-Kastler nets) UMTCs (as in Mac Lane's book)

$$\{I \subset \mathbb{R} \mapsto \mathcal{A}(I)\} \xrightarrow{\text{DHR construction}} \mathrm{DHR}\{\mathcal{A}\}$$

- $I \subset \mathbb{R}$ open bounded intervals, $\mathcal{A}(I) = \mathcal{A}(I)''$ local observables
- Möb $\curvearrowright \mathbb{R}$ and covariantly on $\{\mathcal{A}\}$
- $\exists!$ vacuum vector Ω, split property, Haag duality (on $\mathbb{R}$)
- Rationality = finite number of superselection sectors (positive energy i.e. DHR representations)
  - Examples: Virasoro ($c < 1$) minimal models, $SU(N)$-currents, orbifolds [cf. Marcel's talk], tensor products, finite index extensions

- "objects" = DHR endomorphisms ρ, σ, id, ... of $\{\mathcal{A}\}$
- "arrows" = intertwiners $t : \rho \to \sigma$
- "tensor product" = composition $\rho \times \sigma = \rho\sigma$
- "braiding" = $\mathcal{E}_{\rho,\sigma} : \rho\sigma \to \sigma\rho$ subject to "commutative diagrams"
- Modularity = non-degeneracy condition on the braiding

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Important numerical invariants of $\{\mathcal{A}\}$ can be extracted from $\mathrm{DHR}\{\mathcal{A}\}$:

- statistical dimensions d_ρ and phases ω_ρ of $\rho \in \mathrm{DHR}\{\mathcal{A}\}$
  $\rightsquigarrow$ classification of DHR sectors $[\rho]$ [DHR 71]
- modular matrices S, T
  $\rightsquigarrow$ structure of the DHR category [Rehren 88]

They have more analytic counterparts:

- $d_\rho \sim$ index of subfactors [Longo 89]
- $\omega_\rho \sim$ conformal spin [DHR 74], [Guido, Longo 96]
- S, $T \sim$ modular transformations of Virasoro characters [Verlinde 88]

They depend **only** on the abstract UMTC class of $\mathrm{DHR}\{\mathcal{A}\}$, i.e., different QFTs might give the same numbers, and "the same" category (according to Mac Lane).

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Tensor products are a source of RCFTs: given $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$ let

$$(\mathcal{A} \otimes \mathcal{B})(I) := \mathcal{A}(I) \otimes \mathcal{B}(I) \text{ in } \mathcal{B}(\mathcal{H}^{\mathcal{A}} \otimes \mathcal{H}^{\mathcal{B}}) \quad , \quad \Omega^{\mathcal{A}\otimes\mathcal{B}} := \Omega^{\mathcal{A}} \otimes \Omega^{\mathcal{B}}$$

then

$$\{I \subset \mathbb{R} \mapsto \mathcal{A} \otimes \mathcal{B}(I)\} \longmapsto \mathrm{DHR}\{\mathcal{A} \otimes \mathcal{B}\} \simeq \mathrm{DHR}\{\mathcal{A}\} \boxtimes \mathrm{DHR}\{\mathcal{B}\}$$

$$\mathcal{E}_{\rho\boxtimes\sigma,\tau\boxtimes\eta} := \mathcal{E}^{\mathcal{A}}_{\rho,\tau} \boxtimes \mathcal{E}^{\mathcal{B}}_{\sigma,\eta}$$

In particular, if $\{\mathcal{B}\}$ has no non-vacuum DHR sectors, "holomorphic" RCFTs, i.e., $\mathrm{DHR}\{\mathcal{B}\} \simeq \mathrm{Vec}$, where $[\mathrm{id}] \simeq \mathbb{C}$ and $[\mathrm{id}] \oplus \cdots \oplus [\mathrm{id}] \simeq \mathbb{C}^n$, then

$$\mathrm{DHR}\{\mathcal{A} \otimes \mathcal{B}\} \simeq \mathrm{DHR}\{\mathcal{A}\}$$

because $\mathcal{C} \boxtimes \mathrm{Vec} \simeq \mathcal{C}$ for every $\mathbb{C}$-linear additive category $\mathcal{C}$. However

$$\{\mathcal{A} \otimes \mathcal{B}\} \not\cong \{\mathcal{A}\}$$

unless $\{\mathcal{B}\} \cong \{\mathbb{C}\}$.

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Question: how to complete the DHR construction? [G, Rehren 15]

Observation:

$$\text{if } \rho,\ \sigma \text{ are resp. left/right localizable} \quad \Rightarrow \quad \varepsilon_{\rho,\sigma} = \mathbb{1}$$

- specific feature of the DHR braiding (missing on Mac Lane's book)
- refers to spacetime DHR "localizability" of ρ and σ (here on $\mathbb{R}$)
- DHR endomorphisms often commute $\rho\sigma = \sigma\rho$ (consequence of locality)
- $\varepsilon_{\rho,\sigma} = \mathbb{1}$ + naturality = very definition of DHR braiding

New input: consider the whole realization of $\mathrm{DHR}\{\mathcal{A}\}$ as braided tensor category of endomorphisms of the net, or better, locally:

$\mathcal{A}(I_0)$ fixed local algebra $\sim$ $\mathcal{M}_0$ injective type III_1 factor

$\mathrm{DHR}^{I_0}\{\mathcal{A}\}$ local DHR category $\sim$ $\mathcal{C}$ strict UMTC

$$\mathrm{DHR}^{I_0}\{\mathcal{A}\} \underset{\text{restr.}}{\hookrightarrow} \mathrm{End}(\mathcal{A}(I_0)) \quad \sim \quad \mathcal{C} \hookrightarrow \mathrm{End}(\mathcal{M}_0)$$

$$\rho \longmapsto \rho_{\restriction \mathcal{A}(I_0)}$$
$$t \longmapsto t$$

- strict tensor functor
- faithful, full
- replete image

"braided action" of $\mathrm{DHR}^{I_0}\{\mathcal{A}\}$ on $\mathcal{A}(I_0)$

Isomorphism of nets $\{\mathcal{A}\} \cong \{\mathcal{B}\}$, i.e., $W : \mathcal{H}^{\mathcal{A}} \to \mathcal{H}^{\mathcal{B}}$, $W\mathcal{A}(I)W^* = \mathcal{B}(I)$, $I \subset \mathbb{R}$, and $W\Omega^{\mathcal{A}} = \Omega^{\mathcal{B}}$ gives an isomorphism of braided actions, i.e., invariant for nets:

$$\mathrm{Ad}_W : \mathcal{A}(I_0) \to \mathcal{B}(I_0), \quad \mathrm{Ad}_W \circ \rho^{\mathcal{A}} \circ \mathrm{Ad}_{W^*} = \rho^{\mathcal{B}}, \quad \mathrm{Ad}_W(t^{\mathcal{A}}) = t^{\mathcal{B}}$$

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Why **"braided"**? Forget for a moment the braiding and its realization.

Let $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$ s.t. $\mathrm{DHR}^{I_0}\{\mathcal{A}\} \simeq \mathrm{DHR}^{I_0}\{\mathcal{B}\}$ as abstract tensor categories, i.e.

$$\mathrm{DHR}^{I_0}\{\mathcal{A}\} \xrightarrow[F]{\simeq} \mathrm{DHR}^{I_0}\{\mathcal{B}\}$$

$$\rho^{\mathcal{A}} \longmapsto F(\rho)^{\mathcal{B}}$$
$$t^{\mathcal{A}} \longmapsto F(t)^{\mathcal{B}}$$

$$\begin{array}{ccc} \mathrm{DHR}^{I_0}\{\mathcal{A}\} & \hookrightarrow & \mathrm{End}(\mathcal{A}(I_0)) \\ F \downarrow \wr & & \downarrow F_V \\ \mathrm{DHR}^{I_0}\{\mathcal{B}\} & \hookrightarrow & \mathrm{End}(\mathcal{B}(I_0)) \end{array}$$

tensor $F(\rho) \times F(\sigma) \cong F(\rho \times \sigma)$

(strict tensor if $F(\rho) \times F(\sigma) = F(\rho \times \sigma)$)

- where $F_V(\rho) := \mathrm{Ad}_V \circ \rho \circ \mathrm{Ad}_{V^*}$ "spatial" strict tensor functor [Popa 95], [Izumi 15] s.t.

$$\mathrm{Ad}_V : \mathcal{A}(I_0) \to \mathcal{B}(I_0), \quad F_V(\rho) \cong F(\rho), \quad \mathrm{Ad}_V(t) \cong F(t)$$

  $\Rightarrow$ there is a unique action of $\mathrm{DHR}^{I_0}\{\mathcal{A}\}$ on the injective type III_1 factor, as a tensor category, where the equivalence is realized by F_V

- however F_V need **not** be a braided equivalence: $\mathrm{Ad}_V(\mathcal{E}^{\mathcal{A}}_{\rho,\sigma}) \neq \mathcal{E}^{\mathcal{B}}_{F_V(\rho),F_V(\sigma)}$

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Braided actions & abstract points

The braiding must play a role. How to use the DHR braiding feature: $\varepsilon_{\rho,\sigma} = \mathbb{1}$?

- Consider "abstract points" of $\mathcal{A}(I_0)$:
  - **geometric** point $p \in I_0$ i.e. $I_0 = I_L \cup \{p\} \cup I_R\,,\; I_L < I_R$

    $$\{\mathcal{A}\} \rightsquigarrow \big(\mathcal{A}(I_L)\,, \mathcal{A}(I_R)\,, \mathrm{DHR}^{I_L}\{\mathcal{A}\}\,, \mathrm{DHR}^{I_R}\{\mathcal{A}\}\big)$$

  - **abstract** point $\widehat{p}$ of $\mathcal{A}(I_0)$ w.r.t. $\mathrm{DHR}^{I_0}\{\mathcal{A}\} \hookrightarrow \mathrm{End}(\mathcal{A}(I_0))$

    $$\widehat{p} := \big(\mathcal{N}\,, \mathcal{N}^c\,, \mathcal{C}_{\mathcal{N}}\,, \mathcal{C}_{\mathcal{N}^c}\big) + \text{ conditions}$$

  - "relative commutants" of subalgebras and subcategories, e.g. $\mathcal{N}^c = \mathcal{N}' \cap \mathcal{A}(I_0)$ and $\mathcal{C}^c = \mathcal{C}' \cap \mathrm{DHR}^{I_0}\{\mathcal{A}\}$
  - "duality relations" [Doplicher 82] subalgebras $\leftrightarrow$ subcategories, e.g. $\mathcal{C}_{\mathcal{N}} = (\mathcal{N}^c)^{\perp}$ and $\mathcal{N} = (\mathcal{C}_{\mathcal{N}^c})^{\perp}$

$$\begin{array}{ccc} \mathcal{N} & \overset{\perp}{\longleftrightarrow} & \mathcal{C}_{\mathcal{N}^c} \\ {\scriptstyle c}\updownarrow & & \updownarrow{\scriptstyle c} \\ \mathcal{N}^c & \overset{\perp}{\longleftrightarrow} & \mathcal{C}_{\mathcal{N}} \end{array} \quad , \qquad \underset{\rho \in \mathcal{C}_{\mathcal{N}}\,,\, \sigma \in \mathcal{C}_{\mathcal{N}^c}}{\varepsilon_{\rho,\sigma} = \mathbb{1}}$$

- Algebra replaces geometry:

  $\mathcal{C}_{\mathcal{N}} \hookrightarrow \mathrm{End}(\mathcal{N})$ well defined, $\mathcal{C}_{\mathcal{N}}$ fusion, $\varepsilon_{\rho,\sigma} = \mathbb{1}$ finite system of eqns

- Let $p \in I_0$, $t \mapsto \Lambda^t_{I_0}$ one-parameter group of dilations of I_0, Δ_ω modular operator of $\mathcal{A}(I_0)$ w.r.t. ω, then

$$\mathrm{Ad}_{\Delta^{it}_\omega} : \mathcal{A}(I_0) \to \mathcal{A}(I_0)\,, \quad \Delta^{it}_\omega\,\widehat{p}\,\Delta^{-it}_\omega \text{ again abstract point of } \mathcal{A}(I_0)$$

  - $\omega =$ vacuum, [Bisognano-Wichmann] $\Delta^{it}_\omega\,\widehat{p}\,\Delta^{-it}_\omega = \widehat{q}$, where $q = \Lambda^{-2\pi t}_{I_0}(p)$
  - $\omega =$ any faithful normal state of $\mathcal{A}(I_0)$, [Longo 97] $\rightsquigarrow$ "fuzzy points"

- Similarly, let $u \in \mathcal{U}(\mathcal{A}(I_0))$ unitary group of $\mathcal{A}(I_0)$, then

$$\mathrm{Ad}_u : \mathcal{A}(I_0) \to \mathcal{A}(I_0)\,, \quad u\,\widehat{p}\,u^* \text{ again abstract point of } \mathcal{A}(I_0)$$

  - $u \in \mathcal{U}(\mathcal{A}(I_1)), I_1 \subset I_0$ and $p \notin I_1$, then $u\,\widehat{p}\,u^* = \widehat{p}$
  - otherwise $u\,\widehat{p}\,u^* \neq \widehat{p} \rightsquigarrow$ "fat points"

- Both come from groups of braided tensor autoequivalences of the DHR braided action on $\mathcal{A}(I_0)$

- Let $\{\mathcal{A} \otimes \mathcal{B}\}$ on $\mathbb{R}$, embed "diagonally" $\{\mathcal{A} \otimes \mathcal{B}(I)\} \subset \{\mathcal{A}(I) \otimes \mathcal{B}(J)\}$ on $\mathbb{R}^2$
  If $p_1, p_2 \in I_0$, take $\widehat{p}_1$ in $\mathcal{A}(I_0)$ and $\widehat{p}_2$ in $\mathcal{B}(I_0)$, then

  $$\widehat{p}_1 \otimes \widehat{p}_2 \text{ is an abstract point of } \mathcal{A} \otimes \mathcal{B}(I_0)$$

  geometric in I_0 iff $p_1 = p_2 \rightsquigarrow$ "2D points"

Abstract points lead far away from geometry (can be fuzzy, fat, 2D,...), while geometric points p, q of $\mathbb{R}$ are **totally ordered**: $p \leq q$ or $q \leq p$

- Why looking at abstract points?
  - identify a subfamily of RCFTs which can be classified, "prime" conformal nets (**Idea**: rule out $\otimes$-nets. **Tools**: prime decomposition of UMTCs [Müger 03] + structure of the two-interval inclusion [KLM 01])
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Let $\widehat{p}$, $\widehat{q}$ abstract points, $\widehat{p} = (\mathcal{N}, \mathcal{N}^c, \ldots)$, $\widehat{q} = (\mathcal{M}, \mathcal{M}^c, \ldots)$, of $\mathcal{A}(I_0)$

$\mathcal{N} \vee \mathcal{M}^c \subset (\mathcal{N} \vee \mathcal{M}^c)^{cc}$ "abstract" two-interval inclusion (cf. [KLM 01])

$$\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}} \quad (= \mathrm{DHR}^{\overline{pq}}\{\mathcal{A}\} \text{ if } p, q \in I_0 \text{ and } p < q\,)$$

- Abstract points can be "algebraically compared" by looking at two natural intermediate algebras in $\mathcal{N} \vee \mathcal{M}^c \subset \ldots \subset (\mathcal{N} \vee \mathcal{M}^c)^{cc}$
  - $(\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}})^{\perp}$ = fixed points algebra
  - $\mathcal{U}(\mathcal{N}, \mathcal{M}^c)$ = unitary charge transporters from $\mathcal{C}_{\mathcal{N}}$ to $\mathcal{C}_{\mathcal{M}^c}$

$$\widehat{p} \sim \widehat{q} \quad \text{if} \quad \begin{cases} (\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}})^{\perp} = \mathcal{N} \vee \mathcal{M}^c & (1) \\ \mathcal{U}(\mathcal{N}, \mathcal{M}^c) = (\mathcal{N} \vee \mathcal{M}^c)^{cc} & (2) \end{cases} \quad \text{and} \quad [\mathcal{N} \leftrightarrow \mathcal{M}\,] \quad \begin{matrix} (1)' \\ (2)' \end{matrix}$$

- In the geometric case:
  - (1) conjecture of [Dop 82] in 4D, holds for RCFTs in 1D [GR 15]
  - (2) charge transporters generate relative commutants [KLM 01]

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$$\widehat{p} \sim \widehat{q} \quad \text{if} \quad \begin{cases} (\mathcal{C}_{\mathcal{N}^c} \cap \mathcal{C}_{\mathcal{M}})^\perp = \mathcal{N} \vee \mathcal{M}^c & (1) \\ \mathcal{U}(\mathcal{N}, \mathcal{M}^c) = (\mathcal{N} \vee \mathcal{M}^c)^{cc} & (2) \end{cases} \quad \text{and} \quad [\mathcal{N} \leftrightarrow \mathcal{M}\,] \quad \begin{matrix} (1)' \\ (2)' \end{matrix}$$

- In the geometric case:
  - (1) conjecture of [Dop 82] in 4D, holds for RCFTs in 1D [GR 15]
  - (2) charge transporters generate relative commutants [KLM 01]

- **Main fact**: let $\{\mathcal{A}\}$ prime conformal net, $\boxed{\widehat{p} \sim \widehat{q} \;\Rightarrow\; \widehat{p} \leq \widehat{q} \text{ or } \widehat{q} \leq \widehat{p}}$
  - where $\widehat{p} \leq \widehat{q} := \mathcal{N} \subset \mathcal{M}, \ldots$ "algebraically ordered"
  - essential use of primality and $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$ on both $\widehat{p}$ and $\widehat{q}$

- **Classification** (under two more conditions):
  - $\widehat{p} \sim \widehat{q} \sim \widehat{r} \;\Rightarrow\; \widehat{p} \sim \widehat{r}$ (transitivity, for $\widehat{p} \sim \widehat{q}$ or any $\widehat{p} \approx \widehat{q} \Rightarrow \widehat{p} \sim \widehat{q}$)
  - $\widehat{p}, \widehat{q} \;\Rightarrow\; \widehat{p} = V\widehat{q}V^*$ (unitary equivalence, fixed $\mathrm{DHR}^{I_0}\{\mathcal{A}\}$, in prime nets)

$\Rightarrow$ DHR braided action on a fixed local algebra $\mathrm{DHR}^{I_0}\{\mathcal{A}\} \hookrightarrow \mathrm{End}(\mathcal{A}(I_0))$ completely classifies prime conformal nets

- compute abstract points of $\mathcal{A}(I_0)$ w.r.t $\mathrm{DHR}^{I_0}\{\mathcal{A}\} \hookrightarrow \mathrm{End}(\mathcal{A}(I_0))$
- use previous "main fact"
- use additivity of local algebras + Dedekind's completeness of $\mathbb{R}$
- algebraic Haag's theorem [Weiner 11]

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- Question: how to complete the DHR construction $\{\mathcal{A}\} \longmapsto \mathrm{DHR}\{\mathcal{A}\}$?
(i.e. add more invariants on the DHR side)
- New input: DHR braided action on local algebras $\mathrm{DHR}^{I_0}\{\mathcal{A}\} \hookrightarrow \mathrm{End}(\mathcal{A}(I_0))$
(sits sharply between subfactor theory/tensor categories and QFT)
- Exploit $\mathcal{E}_{\rho,\sigma} = \mathbb{1}$: consider abstract points $\widehat{p}$, compare them $\widehat{p} \sim \widehat{q}$, characterize (using abstract points) prime conformal nets (rule out $\otimes$-nets)
- Aim: reconstruct (not spacetime, but) local algebras $\{\mathcal{A}\}$ inside $\mathcal{A}(I_0)$
using $\widehat{p} \sim \widehat{q} \;\Rightarrow\; \widehat{p} \leq \widehat{q}$ or $\widehat{q} \leq \widehat{p}$
- **Open questions**: find other degeneracies of the DHR construction, find examples of prime conformal nets, improve algebraic conditions, realizability problem for UMTCs