The Bivariant Cuntz Semigroup and Classification of Stably Finite C*-algebras

Gabriele N. Tornetta
joint with J. Bosa and J. Zacharias
School of Mathematics and Statistics
University of Glasgow

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Outline

Introduction

A von Neumann algebra is generated, as a Banach space, by the set of its projections.

Part of the structure of a von Neumann algebra M can then be inferred by comparing its projections.

For $p, q \in M$ projections, we say that p and q are Murray-von Neumann equivalent (in symbols $p \sim q$) if there is a partial isometry $v \in M$ such that $p = v^*v$ and $q = vv^*$.

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Let $M_\infty(M)$ denote the infinite matrix algebra with entries in the von Neumann algebra M and let $P(M_\infty(M))$ denote the set of its projections.

The Murray-von Neumann semigroup of M is defined as

$$V(M) := P(M_\infty(M))/\sim .$$

A von Neumann algebra is a special type of C*-algebra. It is easy to see that the definition of Murray-von Neumann equivalence can be extended to the latter. For a C*-algebra A one can then set

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The Murray-von Neumann semigroup is the starting point of K-theory for operator algebras. For a unital C*-algebra A one defines

$$K_0(A) := \Gamma(V(A)).$$

Examples.

$$V(\mathbb{C}) \cong V(M_n) \cong V(K) \cong \mathbb{N}_0$$

More generally $V(A \otimes K) \cong V(A)$, i.e. V is a stable functor.

$$K_0(\mathbb{C}) \cong K_0(M_n) \cong K_0(K) \cong \mathbb{Z} \qquad \text{and} \qquad K_0(A \otimes K) \cong K_0(A).$$

$$V(B(\ell^2(\mathbb{N}))) \cong \mathbb{N}_0 \sqcup \{\infty\}$$

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The Cuntz Semigroup

Contrary to a von Neumann algebra, a C*-algebra need not have any projections.

Example. $V(C_0(X)) \cong \{0\}$ for a connected, locally compact, non-compact Hausdorff space X.

Instead of comparing projections one can then compare positive elements.

Let A be a C*-algebra and $a, b \in A^+$ be positive elements. Say that a is Cuntz-below (in symbols $a \precsim b$) if there exists $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that

$$\lim_{n\to\infty} \|x_n^* b x_n - a\| = 0.$$

Say that $a \sim b$ if both $a \precsim b$ and $b \precsim a$ hold. In this case a and b are said to be *Cuntz equivalent*.

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Cuntz originally defined the Cuntz semigroup as the set of classes

$$W(A) := M_\infty(A)^+/\sim$$

endowed with the binary operation $+$ coming from direct sum

$$[a] + [b] := [a \oplus b].$$

However the functor W is not stable nor continuous under inductive limits.

Coward, Elliott and Ivanescu proposed a new definition of the Cuntz semigroup that is stable and preserves sequential inductive limits

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Kasparov found a way to "unite" K-theory and K-homology into a unique bifunctor: KK-theory. Indeed, $KK(\mathbb{C}, B) \cong K_0(B)$ for any (graded) C*-algebra B.

The Bivariant Cuntz Semigroup is an attempt to provide an analogue of the above relation for the the Cuntz theory of comparison of positive elements.

The starting point is the key observation of Winter and Zacharias that every c.p. order zero map (i.e. orthogonality preserving, completely positive and linear) induces a map at the level of the Cuntz semigroups by functoriality.

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For any two c.p.c. order zero maps $\phi, \psi : A \to B$, we say that ϕ is Cuntz-below ψ (in symbols $\phi \precsim \psi$) if there exists $\{x_n\}_{n\in\mathbb{N}} \subset B$ such that

$$\lim_{n\to\infty} \|x_n^*\psi(a)x_n - \phi(a)\| = 0, \qquad \forall a \in A.$$

We say that ϕ and ψ are Cuntz equivalent (in symbols $\phi \sim \psi$) if both $\phi \precsim \psi$ and $\psi \precsim \phi$ hold.

We then define the bivariant Cuntz semigroup $\mathrm{Cu}(A, B)$ of a pair of C*-algebras A and B as the set of equivalence classes of c.p.c. order zero maps $\phi : A \otimes K \to B \otimes K$. The binary operation is given by "direct sum" of maps

$$[\phi] + [\psi] := [\phi \,\hat{\oplus}\, \psi]$$

where $(\phi \,\hat{\oplus}\, \psi)(a) := \phi(a) \oplus \psi(a)$ for any $a \in A \otimes K$.

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Theorem. $\mathrm{Cu}(A \otimes K, B \otimes K) \cong \mathrm{Cu}(A, B)$ for any pair of C*-algebras A and B.

Claim. $\mathrm{Cu}(\mathbb{C}, B) \cong \mathrm{Cu}(B)$ for any C*-algebra B.

Proof. Any class $\Phi \in \mathrm{Cu}(A, B)$ has a representative of the form $\phi \otimes \mathrm{id}_K$, where $\phi : A \to B \otimes K$ c.p.c. order zero. When $A = \mathbb{C}$ we have $\phi(z) = zh$ for any $z \in \mathbb{C}$, where $h = \phi(1) \in (B \otimes K)^+$. Hence Φ can be identified with $[h]$ inside $\mathrm{Cu}(B)$. □

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The bivariant Cuntz semigroup and KK-theory share the following properties.

- Additivity in both arguments
- Countable additivity in the first argument
- Functoriality
- Stability in both arguments
- No general continuity under C*-inductive limits
- **Composition product**

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Classification

There is an analogue of Kasparov's product for the bivariant Cuntz semigroup

$$\cdot\ : \mathrm{Cu}(A,B) \times \mathrm{Cu}(B,C) \to \mathrm{Cu}(A,C)$$
$$[\phi]\cdot[\psi] := [\psi \circ \phi].$$

This is well-defined, since composition of c.p.c. order zero maps yields a c.p.c. order zero map and $\cdot$ is "compatible" with $\precsim$.

For any C*-algebra A, $\mathrm{Cu}(A,A)$ is a semiring with unit $\iota_A := [\mathrm{id}_{A\otimes K}]$.

Definition. An element $\Phi \in \mathrm{Cu}(A,B)$ is invertible if there exists $\Psi \in \mathrm{Cu}(B,A)$ such that

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This is well-defined, since composition of c.p.c. order zero maps yields a c.p.c. order zero map and $\cdot$ is "compatible" with $\precsim$.

For any C*-algebra A, $\mathrm{Cu}(A, A)$ is a semiring with unit $\iota_A := [\mathrm{id}_{A \otimes K}]$.

Definition. An element $\Phi \in \mathrm{Cu}(A, B)$ is invertible if there exists $\Psi \in \mathrm{Cu}(B, A)$ such that

$$\Phi \cdot \Psi = \iota_A \qquad \text{and} \qquad \Psi \cdot \Phi = \iota_B.$$

Classification

Definition. Two C*-algebras A and B are Cu-equivalent if there exists an invertible element in $\mathrm{Cu}(A, B)$.

Cu-equivalence is not strong enough to capture isomorphism between C*-algebras.

Example. Consider $\mathrm{Cu}(M_n, M_m) \cong \mathbb{N}_0 \sqcup \{\infty\}$ for any $n, m \in \mathbb{N}$. Clearly $1 \in \mathbb{N}_0 \sqcup \{\infty\}$ is an invertible element, but $M_n \ncong M_m$ unless $m = n$. However $M_n \otimes K \cong M_m \otimes K$ for every $m, n \in \mathbb{N}$.

Standard classification results (Elliott, Kirchberg-Phillips, ...) suggest that a scale condition to strengthen Cu-equivalence is needed.

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Classification

Definition. Two C^*-algebras A and B are Cu-equivalent if there exists an invertible element in $\mathrm{Cu}(A, B)$.

Cu-equivalence is not strong enough to capture isomorphism between C^*-algebras.

Example. Consider $\mathrm{Cu}(M_n, M_m) \cong \mathbb{N}_0 \sqcup \{\infty\}$ for any $n, m \in \mathbb{N}$. Clearly $1 \in \mathbb{N}_0 \sqcup \{\infty\}$ is an invertible element, but $M_n \ncong M_m$ unless $m = n$. However $M_n \otimes K \cong M_m \otimes K$ for every $m, n \in \mathbb{N}$.

Standard classification results (Elliott, Kirchberg-Phillips, ...) suggest that a scale condition to strengthen Cu-equivalence is needed.

Classification

Definition. An element $\Phi \in \mathrm{Cu}(A, B)$ is strictly invertible if there exist c.p.c. order zero maps $\phi : A \to B$ and $\psi : B \to A$ such that

i. $[\phi \otimes \mathrm{id}_K] = \Phi$;

ii. $\psi \circ \phi \sim \mathrm{id}_A$ and $\phi \circ \psi \sim \mathrm{id}_B$.

Condition ii. is equivalent to: $[\phi \otimes \mathrm{id}_K] \cdot [\psi \otimes \mathrm{id}_K] = \iota_A$ and $[\psi \otimes \mathrm{id}_K] \cdot [\phi \otimes \mathrm{id}_A] = \iota_K$.

By defining the scale of $\mathrm{Cu}(A, B)$ as $\Sigma(\mathrm{Cu}(A, B)) := \{[\phi \otimes \mathrm{id}_K] \in \mathrm{Cu}(A, B) \mid \phi : A \to B \text{ c.p.c. order zero}\}$, a strictly invertible element is invertible and in $\Sigma(\mathrm{Cu}(A, B))$, with inverse in $\Sigma(\mathrm{Cu}(B, A))$

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Every isomorphism $\pi : A \to B$ induces a strictly invertible element in $\mathrm{Cu}(A, B)$.

Classification

Example. Let $n, m \in \mathbb{N}$. If $n < m$ then there are no c.p.c. order zero maps $\phi : M_m \to M_n$. For $n = m$ we can take $\phi = \mathrm{id}_{M_n}$.

A unital C*-algebra is finite if every isometry is a unitary. A unital C*-algebra is stably finite if $(A \otimes K)^\sim$ is finite.

$\mathbb{C}$, M_n, K, M_{2^∞}, AF algebras, AI algebras, ... are all examples of stably finite C*-algebras.

Theorem. Two unital and stably finite C*-algebras A and B are isomorphic if and only if there is a strictly invertible element in $\mathrm{Cu}(A, B)$.

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Thank you.

g.tornetta.1@research.gla.ac.uk
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