Derivation of Maxwell's equations from non-relativistic QED

Nikolai Leopold, Peter Pickl

TUM, May 27^{th} 2016

Outline of the talk

- Motivation
- Pauli-Fierz Hamiltonian/Hartree-Maxwell system
- Main theorem
- Idea of the proof
- Remarks and Outline

Motivation

Question: *Is it possible to derive Maxwell's equations from Quantum electrodynamics?*

Physicists look at Heisenberg equations of the field operators. More rigorous: find a physical situation which gives Maxwell's equations in some limit.

$$
\begin{array}{ccccccccc}
|\alpha\rangle\langle\alpha| & \xleftarrow{N\to\infty} & \gamma_N^{(0,1)} & \longleftarrow & \Psi_N & \longrightarrow & \gamma_N^{(1,0)} & \xrightarrow{N\to\infty} & |\varphi\rangle\langle\varphi| \\
\scriptstyle eff. \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \scriptstyle eff. \\
|\alpha_t\rangle\langle\alpha_t| & \xleftarrow[N\to\infty]{} & \gamma_{N,t}^{(0,1)} & \longleftarrow & \Psi_{N,t} & \longrightarrow & \gamma_{N,t}^{(1,0)} & \xrightarrow[N\to\infty]{} & |\varphi_t\rangle\langle\varphi_t|
\end{array}
$$

Question: *Is it possible to derive Maxwell's equations from Quantum electrodynamics?*

Physicists look at Heisenberg equations of the field operators.
More rigorous: find a physical situation which gives Maxwell's equations in some limit.

$$
\begin{array}{ccccccccc}
|\alpha\rangle\langle\alpha| & \xleftarrow{N\to\infty} & \gamma_N^{(0,1)} & \longleftarrow & \Psi_N & \longrightarrow & \gamma_N^{(1,0)} & \xrightarrow{N\to\infty} & |\varphi\rangle\langle\varphi| \\
eff.\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow eff. \\
|\alpha_t\rangle\langle\alpha_t| & \xleftarrow[N\to\infty]{} & \gamma_{N,t}^{(0,1)} & \longleftarrow & \Psi_{N,t} & \longrightarrow & \gamma_{N,t}^{(1,0)} & \xrightarrow[N\to\infty]{} & |\varphi_t\rangle\langle\varphi_t|
\end{array}
$$

Motivation

Question: *Is it possible to derive Maxwell's equations from Quantum electrodynamics?*

Physicists look at Heisenberg equations of the field operators.
More rigorous: find a physical situation which gives Maxwell's equations in some limit.

$$
\begin{array}{ccccccccc}
|\alpha\rangle\langle\alpha| & \xleftarrow{N\to\infty} & \gamma_N^{(0,1)} & \longleftarrow & \Psi_N & \longrightarrow & \gamma_N^{(1,0)} & \xrightarrow{N\to\infty} & |\varphi\rangle\langle\varphi| \\
{\scriptstyle eff.}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle eff.} \\
|\alpha_t\rangle\langle\alpha_t| & \xleftarrow[N\to\infty]{} & \gamma_{N,t}^{(0,1)} & \longleftarrow & \Psi_{N,t} & \longrightarrow & \gamma_{N,t}^{(1,0)} & \xrightarrow[N\to\infty]{} & |\varphi_t\rangle\langle\varphi_t|
\end{array}
$$

Question: *Is it possible to derive Maxwell's equations from Quantum electrodynamics?*

Physicists look at Heisenberg equations of the field operators.
More rigorous: find a physical situation which gives Maxwell's equations in some limit.

$$\begin{array}{ccccccccc}
|\alpha\rangle\langle\alpha| & \xleftarrow{N\to\infty} & \gamma_N^{(0,1)} & \longleftarrow & \Psi_N & \longrightarrow & \gamma_N^{(1,0)} & \xrightarrow{N\to\infty} & |\varphi\rangle\langle\varphi| \\
{\scriptstyle eff.}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle eff.} \\
|\alpha_t\rangle\langle\alpha_t| & \xleftarrow[N\to\infty]{} & \gamma_{N,t}^{(0,1)} & \longleftarrow & \Psi_{N,t} & \longrightarrow & \gamma_{N,t}^{(1,0)} & \xrightarrow[N\to\infty]{} & |\varphi_t\rangle\langle\varphi_t|
\end{array}$$

Question: *Is it possible to derive Maxwell's equations from Quantum electrodynamics?*

Physicists look at Heisenberg equations of the field operators.
More rigorous: find a physical situation which gives Maxwell's equations in some limit.

$$
\begin{array}{ccccccccc}
|\alpha\rangle\langle\alpha| & \xleftarrow{N\to\infty} & \gamma_N^{(0,1)} & \longleftarrow & \Psi_N & \longrightarrow & \gamma_N^{(1,0)} & \xrightarrow{N\to\infty} & |\varphi\rangle\langle\varphi| \\
\textit{eff.}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\textit{eff.} \\
|\alpha_t\rangle\langle\alpha_t| & \xleftarrow[N\to\infty]{} & \gamma_{N,t}^{(0,1)} & \longleftarrow & \Psi_{N,t} & \longrightarrow & \gamma_{N,t}^{(1,0)} & \xrightarrow[N\to\infty]{} & |\varphi_t\rangle\langle\varphi_t|
\end{array}
$$

spinless Pauli-Fierz Hamiltonian

$$H_N^{PF} := \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}} \right)^2 + \frac{1}{N} \sum_{1 \le j < k \le N} v(x_j - x_k) + H_f ,$$

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda) \right) .$$

- two types of particles: (non-relativistic) charged bosons and photons,
- photons have two polarizations $\epsilon_1(k), \epsilon_2(k)$ $(\nabla \cdot \hat{\boldsymbol{A}}_\kappa = 0)$,
- $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}_p = L^2(\mathbb{R}^{3N}) \otimes [\oplus_{n=0}^{\infty} (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)^{\otimes_s^n}]$,
- scaling: kinetic and potential energy are of the same order.

spinless Pauli-Fierz Hamiltonian

$$H_N^{PF} := \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}} \right)^2 + \frac{1}{N} \sum_{1 \le j < k \le N} v(x_j - x_k) + H_f,$$

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}} \epsilon_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda) \right).$$

- two types of particles: (non-relativistic) charged bosons and photons,
- photons have two polarizations $\epsilon_1(k), \epsilon_2(k)$ $(\nabla \cdot \hat{\boldsymbol{A}}_\kappa = 0)$,
- $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}_p = L^2(\mathbb{R}^{3N}) \otimes [\oplus_{n=0}^{\infty} (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)^{\otimes_s^n}]$,
- scaling: kinetic and potential energy are of the same order.

spinless Pauli-Fierz Hamiltonian

$$H_N^{PF} := \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}} \right)^2 + \frac{1}{N} \sum_{1 \le j < k \le N} v(x_j - x_k) + H_f \, ,$$

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}} \epsilon_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda) \right) .$$

- two types of particles: (non-relativistic) charged bosons and photons,
- photons have two polarizations $\epsilon_1(k), \epsilon_2(k)$ $(\nabla \cdot \hat{\boldsymbol{A}}_\kappa = 0)$,
- $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}_p = L^2(\mathbb{R}^{3N}) \otimes [\oplus_{n=0}^{\infty} (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)^{\otimes_s^n}]$,
- scaling: kinetic and potential energy are of the same order.

spinless Pauli-Fierz Hamiltonian

$$H_N^{PF} := \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}} \right)^2 + \frac{1}{N} \sum_{1 \le j < k \le N} v(x_j - x_k) + H_f \,,$$

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda) \right) .$$

- two types of particles: (non-relativistic) charged bosons and photons,
- photons have two polarizations $\boldsymbol{\epsilon}_1(k), \boldsymbol{\epsilon}_2(k)$ $(\nabla \cdot \hat{\boldsymbol{A}}_\kappa = 0)$,
- $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}_p = L^2(\mathbb{R}^{3N}) \otimes [\oplus_{n=0}^{\infty} (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)^{\otimes_s^n}]$,
- scaling: kinetic and potential energy are of the same order.

$$H_N^{PF} := \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}} \right)^2 + \frac{1}{N} \sum_{1 \le j < k \le N} v(x_j - x_k) + H_f ,$$

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda) \right) .$$

- two types of particles: (non-relativistic) charged bosons and photons,
- photons have two polarizations $\boldsymbol{\epsilon}_1(k), \boldsymbol{\epsilon}_2(k)$ $(\nabla \cdot \hat{\boldsymbol{A}}_\kappa = 0)$,
- $\mathcal{H} = L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}_p = L^2(\mathbb{R}^{3N}) \otimes [\oplus_{n=0}^{\infty} (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)^{\otimes_s^n}]$,
- scaling: kinetic and potential energy are of the same order.

Regime

The scaling can be motivated by the Ehrenfest equations of the field operators:

$$d_t \langle\!\langle \Psi_N, \frac{\hat{\boldsymbol{A}}_\kappa(y)}{\sqrt{N}} \Psi_N \rangle\!\rangle = -\langle\!\langle \Psi_N, \frac{\hat{\boldsymbol{E}}_\kappa(y)}{\sqrt{N}} \Psi_N \rangle\!\rangle,$$

$$d_t \langle\!\langle \Psi_N, \frac{\hat{\boldsymbol{E}}_\kappa(y)}{\sqrt{N}} \Psi_N \rangle\!\rangle = -\frac{2}{N} \sum_{j=1}^{N} \langle\!\langle \Psi_N, \mathbf{e}^i \delta_{il}^{\Lambda}(y - x_j) \left(-i\nabla_j^l - \frac{\hat{\boldsymbol{A}}_\kappa^l(x_j)}{\sqrt{N}} \right) \Psi_N \rangle\!\rangle + \dots$$

Emergence of the effective description:

- $\Psi_N \approx \prod_{i=1}^N \varphi(x_i) \otimes |b\rangle_{\mathcal{F}}$,
- $\langle b, \frac{\hat{\boldsymbol{A}}_\kappa(x)}{\sqrt{N}} b\rangle_{\mathcal{F}}, \langle b, \frac{\hat{\boldsymbol{A}}_\kappa^2(x)}{N} b\rangle_{\mathcal{F}} \to \boldsymbol{A}_\kappa(x,t), \boldsymbol{A}_\kappa^2(x,t)$.

Regime

The scaling can be motivated by the Ehrenfest equations of the field operators:

$$d_t \langle\!\langle \Psi_N, \frac{\hat{\boldsymbol{A}}_\kappa(y)}{\sqrt{N}} \Psi_N \rangle\!\rangle = -\langle\!\langle \Psi_N, \frac{\hat{\boldsymbol{E}}_\kappa(y)}{\sqrt{N}} \Psi_N \rangle\!\rangle,$$

$$d_t \langle\!\langle \Psi_N, \frac{\hat{\boldsymbol{E}}_\kappa(y)}{\sqrt{N}} \Psi_N \rangle\!\rangle = -\frac{2}{N} \sum_{j=1}^{N} \langle\!\langle \Psi_N, \mathbf{e}^i \delta_{il}^{\Lambda}(y - x_j) \left(-i\nabla_j^l - \frac{\hat{\boldsymbol{A}}_\kappa^l(x_j)}{\sqrt{N}} \right) \Psi_N \rangle\!\rangle + \dots$$

Emergence of the effective description:

- $\Psi_N \approx \prod_{i=1}^N \varphi(x_i) \otimes |b\rangle_{\mathcal{F}}$,
- $\langle b, \frac{\hat{\boldsymbol{A}}_\kappa(x)}{\sqrt{N}} b\rangle_{\mathcal{F}}, \langle b, \frac{\hat{\boldsymbol{A}}_\kappa^2(x)}{N} b\rangle_{\mathcal{F}} \to \boldsymbol{A}_\kappa(x,t), \boldsymbol{A}_\kappa^2(x,t)$.

Hartree-Maxwell system of equations

$$\begin{cases} i\partial_t \varphi(t) &= H^{HM}\varphi(t), \\ \nabla \cdot \boldsymbol{A}_\kappa(t) &= 0, \\ \partial_t \boldsymbol{A}_\kappa(t) &= -\boldsymbol{E}_\kappa(t), \\ \partial_t \boldsymbol{E}_\kappa(t) &= -\left(\Delta \boldsymbol{A}_\kappa\right)(t) - \mathbf{e}^i \left(\delta_{il}^{\Lambda} \star \boldsymbol{j}^l\right)(t), \end{cases}$$

where

$$H^{HM} = \left(-i\nabla - \boldsymbol{A}_\kappa\right)^2 + \left(v \star |\varphi|^2\right),$$
$$\boldsymbol{j} = 2\left(\mathrm{Im}(\varphi^* \nabla \varphi) - |\varphi|^2 \boldsymbol{A}_\kappa\right).$$

$$\boldsymbol{A}_\kappa(x,t) = \sum_{\lambda=1,2} \int d^3k \frac{\hat{\kappa}(k)}{\sqrt{2|k|}} \epsilon_\lambda(k) \left(e^{ikx}\alpha(k,\lambda,t) + e^{-ikx}\bar{\alpha}(k,\lambda,t)\right).$$

$$\begin{cases} i\partial_t \varphi(t) &= \left[\left(-i\nabla - \boldsymbol{A}_\kappa\right)^2 + \left(v \star |\varphi|^2\right)\right]\varphi(t), \\ i\partial_t \alpha(k,\lambda,t) &= |k|\alpha(k,\lambda,t) - (2\pi)^{\frac{3}{2}} \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}} \epsilon_\lambda^l(k) \tilde{\boldsymbol{j}}^l(k). \end{cases}$$

Hartree-Maxwell system of equations

$$\begin{cases} i\partial_t \varphi(t) &= H^{HM}\varphi(t), \\ \nabla \cdot \boldsymbol{A}_\kappa(t) &= 0, \\ \partial_t \boldsymbol{A}_\kappa(t) &= -\boldsymbol{E}_\kappa(t), \\ \partial_t \boldsymbol{E}_\kappa(t) &= -\left(\Delta \boldsymbol{A}_\kappa\right)(t) - \mathbf{e}^i \left(\delta_{il}^\Lambda \star \boldsymbol{j}^l\right)(t), \end{cases}$$

where

$$H^{HM} = (-i\nabla - \boldsymbol{A}_\kappa)^2 + \left(v \star |\varphi|^2\right),$$

$$\boldsymbol{j} = 2\left(\mathrm{Im}(\varphi^* \nabla\varphi) - |\varphi|^2 \boldsymbol{A}_\kappa\right).$$

$$\boldsymbol{A}_\kappa(x,t) = \sum_{\lambda=1,2} \int d^3k \frac{\hat{\kappa}(k)}{\sqrt{2|k|}} \epsilon_\lambda(k) \left(e^{ikx}\alpha(k,\lambda,t) + e^{-ikx}\bar{\alpha}(k,\lambda,t)\right).$$

$$\begin{cases} i\partial_t \varphi(t) &= \left[(-i\nabla - \boldsymbol{A}_\kappa)^2 + \left(v \star |\varphi|^2\right)\right]\varphi(t), \\ i\partial_t \alpha(k,\lambda,t) &= |k|\alpha(k,\lambda,t) - (2\pi)^{\frac{3}{2}} \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}} \epsilon_\lambda^l(k) \tilde{\boldsymbol{j}}^l(k). \end{cases}$$

Hartree-Maxwell system of equations

$$
\begin{cases}
i\partial_t \varphi(t) & = H^{HM}\varphi(t), \\
\nabla \cdot \boldsymbol{A}_\kappa(t) & = 0, \\
\partial_t \boldsymbol{A}_\kappa(t) & = -\boldsymbol{E}_\kappa(t), \\
\partial_t \boldsymbol{E}_\kappa(t) & = -\left(\Delta \boldsymbol{A}_\kappa\right)(t) - \mathbf{e}^i \left(\delta_{il}^{\Lambda} \star \boldsymbol{j}^l\right)(t),
\end{cases}
$$

where

$$
H^{HM} = (-i\nabla - \boldsymbol{A}_\kappa)^2 + \left(v \star |\varphi|^2\right),
$$

$$
\boldsymbol{j} = 2\left(\mathrm{Im}(\varphi^* \nabla \varphi) - |\varphi|^2 \boldsymbol{A}_\kappa\right).
$$

$$
\boldsymbol{A}_\kappa(x,t) = \sum_{\lambda=1,2} \int d^3k \, \frac{\hat{\kappa}(k)}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) \left(e^{ikx}\alpha(k,\lambda,t) + e^{-ikx}\bar{\alpha}(k,\lambda,t)\right).
$$

$$
\begin{cases}
i\partial_t \varphi(t) & = \left[(-i\nabla - \boldsymbol{A}_\kappa)^2 + \left(v \star |\varphi|^2\right)\right]\varphi(t), \\
i\partial_t \alpha(k,\lambda,t) & = |k|\alpha(k,\lambda,t) - (2\pi)^{\frac{3}{2}} \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda^l(k) \tilde{\boldsymbol{j}}^l(k).
\end{cases}
$$

Reduced one-particle density matrices:

$$\gamma_N^{(1,0)} := \mathrm{Tr}_{2,\dots,N} \otimes \mathrm{Tr}_{\mathcal{F}} |\Psi_N\rangle\langle\Psi_N|,$$
$$\gamma_N^{(0,1)}(k,\lambda;k',\lambda') := N^{-1} \langle\!\langle \Psi_N, a^*(k',\lambda') a(k,\lambda) \Psi_N \rangle\!\rangle.$$

Theorem

Let $\varphi(x,0) \in H^3(\mathbb{R}^3)$, $\alpha(k,\lambda,0) = 0$, $\Psi_N(0) = \prod_{i=1}^N \varphi(x_i) \otimes |0\rangle_{\mathcal{F}}$, and $v(x) = \frac{1}{|x|}$. Then, for any $t > 0$ there exists a constant $C(t,\Lambda)$ such that

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)} |\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| \leq \frac{C}{\sqrt{N}},$$
$$\mathrm{Tr}_{L^2(\mathbb{R}^3)\otimes\mathbb{C}^2} |\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|| \leq \frac{C}{\sqrt{N}}.$$

Remark: In general, this holds for a larger class of potentials and initial states.

Main theorem

Reduced one-particle density matrices:

$$\gamma_N^{(1,0)} := \mathrm{Tr}_{2,\dots,N} \otimes \mathrm{Tr}_{\mathcal{F}} |\Psi_N\rangle\langle\Psi_N|,$$
$$\gamma_N^{(0,1)}(k,\lambda;k',\lambda') := N^{-1}\langle\!\langle \Psi_N, a^*(k',\lambda')a(k,\lambda)\Psi_N\rangle\!\rangle.$$

Theorem

Let $\varphi(x,0) \in H^3(\mathbb{R}^3)$, $\alpha(k,\lambda,0) = 0$, $\Psi_N(0) = \prod_{i=1}^N \varphi(x_i) \otimes |0\rangle_{\mathcal{F}}$, and $v(x) = \frac{1}{|x|}$. Then, for any $t > 0$ there exists a constant $C(t,\Lambda)$ such that

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)} |\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| \leq \frac{C}{\sqrt{N}},$$
$$\mathrm{Tr}_{L^2(\mathbb{R}^3)\otimes\mathbb{C}^2} |\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|| \leq \frac{C}{\sqrt{N}}.$$

Remark: In general, this holds for a larger class of potentials and initial states.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$
Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$
Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$
Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$
Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$
Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$
Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$
Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$
Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$
Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$

Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$

Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$

Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$

Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$

Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$

Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$
Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$
Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$
Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$
Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$
Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$
Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$
Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$
Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$
Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$
Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$
Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$
Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Idea of the proof

Introduce the functional: $\beta(t) = \beta^a(t) + \beta^b(t) + \beta^c(t) + \beta^d(t)$.

- β^a measures if the charged bosons are in a condensate,
- β^b and β^c measure if the photons are close to a coherent state,
- β^d restricts the class of Many-body initial states.

Initially: $\beta(0) \approx 0$
Show: $d_t\beta(t) \leq C(\beta(t) + \frac{1}{N})$
Grönwall: $\beta(t) \leq e^{Ct}\left(\beta(0) + \frac{Ct}{N}\right)$
Tasks of $\beta(t)$:

- $\beta(0) \approx 0$ defines conditions on the initial states: $(\Psi_N(0), \varphi(0), \alpha(0))$.
- $\beta(t)$ is a measure of condensation at later times.

Let $\varphi \in L^2(\mathbb{R}^3)$ and $p_j : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ be given by

$$f(x_1, \ldots, x_N) \mapsto \varphi(x_j) \int d^3x_j \; \varphi^*(x_j) f(x_1, \ldots, x_N).$$

Define $q_j := 1 - p_j$ and the functional

$$\beta^a[\Psi_N, \varphi] := N^{-1} \sum_{j=1}^{N} \langle\langle \Psi_N, q_j \otimes \mathbb{1}_{\mathcal{F}} \Psi_N \rangle\rangle.$$

β^a measures the relative number of particles which are not in the state φ:

- $\beta^a \leq \mathrm{Tr}_{L^2(\mathbb{R}^3)} |\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|| \leq C\sqrt{\beta^a}$,
- $\Psi_N = \prod_{j=1}^{N} \varphi(x_j, t) \otimes |0\rangle_{\mathcal{F}} \Rightarrow \beta^a[\Psi_N, \varphi] = 0$.

Let $\varphi \in L^2(\mathbb{R}^3)$ and $p_j : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ be given by

$$f(x_1, \ldots, x_N) \mapsto \varphi(x_j) \int d^3x_j \; \varphi^*(x_j) f(x_1, \ldots, x_N).$$

Define $q_j := 1 - p_j$ and the functional

$$\beta^a[\Psi_N, \varphi] := N^{-1} \sum_{j=1}^{N} \langle\langle \Psi_N, q_j \otimes \mathbb{1}_{\mathcal{F}} \Psi_N \rangle\rangle.$$

β^a measures the relative number of particles which are not in the state φ:

- $\beta^a \leq \mathrm{Tr}_{L^2(\mathbb{R}^3)} |\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|| \leq C\sqrt{\beta^a}$,
- $\Psi_N = \prod_{j=1}^{N} \varphi(x_j, t) \otimes |0\rangle_{\mathcal{F}} \Rightarrow \beta^a[\Psi_N, \varphi] = 0$.

Let $\varphi \in L^2(\mathbb{R}^3)$ and $p_j : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ be given by

$$f(x_1, \ldots, x_N) \mapsto \varphi(x_j) \int d^3x_j \; \varphi^*(x_j) f(x_1, \ldots, x_N).$$

Define $q_j := 1 - p_j$ and the functional

$$\beta^a[\Psi_N, \varphi] := N^{-1} \sum_{j=1}^{N} \langle\!\langle \Psi_N, q_j \otimes \mathbb{1}_{\mathcal{F}} \Psi_N \rangle\!\rangle.$$

β^a measures the relative number of particles which are not in the state φ:

- $\beta^a \leq \mathrm{Tr}_{L^2(\mathbb{R}^3)} |\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|| \leq C\sqrt{\beta^a}$,
- $\Psi_N = \prod_{j=1}^{N} \varphi(x_j, t) \otimes |0\rangle_{\mathcal{F}} \Rightarrow \beta^a[\Psi_N, \varphi] = 0$.

β^b and β^c

$$\beta^b[\Psi_N,\alpha] := \int d^3y \langle\!\langle \Psi_N, \left(\frac{\hat{\boldsymbol{A}}_\kappa^-(y)}{\sqrt{N}} - \boldsymbol{A}_\kappa^-(y,t) \right) \left(\frac{\hat{\boldsymbol{A}}_\kappa^+(y)}{\sqrt{N}} - \boldsymbol{A}_\kappa^+(y,t) \right) \Psi_N \rangle\!\rangle,$$

$$\beta^c[\Psi_N,\alpha] := \int d^3y \langle\!\langle \Psi_N, \left(\frac{\hat{\boldsymbol{E}}_\kappa^-(y)}{\sqrt{N}} - \boldsymbol{E}_\kappa^-(y,t) \right) \left(\frac{\hat{\boldsymbol{E}}_\kappa^+(y)}{\sqrt{N}} - \boldsymbol{E}_\kappa^+(y,t) \right) \Psi_N \rangle\!\rangle.$$

β^b and β^c measure the fluctuations of $\hat{\boldsymbol{A}}_\kappa$ and $\hat{\boldsymbol{E}}_\kappa$ around $\boldsymbol{A}_\kappa$ and $\boldsymbol{E}_\kappa$:

- $\mathrm{Tr}_{L^2(\mathbb{R}^3)\otimes\mathbb{C}^2} |\gamma_N^{(0,1)} - |\alpha\rangle\langle\alpha|| \leq C\sqrt{\beta^b + \beta^c}$,
- $\Psi_N = \prod_{j=1}^N \varphi(x_j,t) \otimes |0\rangle_{\mathcal{F}} \wedge \alpha = 0 \Rightarrow \beta^b = \beta^c = 0$.

β^b and β^c

$$\beta^b[\Psi_N,\alpha] := \int d^3y \langle\!\langle \Psi_N, \left(\frac{\hat{\boldsymbol{A}}_\kappa^-(y)}{\sqrt{N}} - \boldsymbol{A}_\kappa^-(y,t) \right) \left(\frac{\hat{\boldsymbol{A}}_\kappa^+(y)}{\sqrt{N}} - \boldsymbol{A}_\kappa^+(y,t) \right) \Psi_N \rangle\!\rangle,$$

$$\beta^c[\Psi_N,\alpha] := \int d^3y \langle\!\langle \Psi_N, \left(\frac{\hat{\boldsymbol{E}}_\kappa^-(y)}{\sqrt{N}} - \boldsymbol{E}_\kappa^-(y,t) \right) \left(\frac{\hat{\boldsymbol{E}}_\kappa^+(y)}{\sqrt{N}} - \boldsymbol{E}_\kappa^+(y,t) \right) \Psi_N \rangle\!\rangle.$$

β^b and β^c measure the fluctuations of $\hat{\boldsymbol{A}}_\kappa$ and $\hat{\boldsymbol{E}}_\kappa$ around $\boldsymbol{A}_\kappa$ and $\boldsymbol{E}_\kappa$:

- $\mathrm{Tr}_{L^2(\mathbb{R}^3)\otimes\mathbb{C}^2} |\gamma_N^{(0,1)} - |\alpha\rangle\langle\alpha|| \leq C\sqrt{\beta^b + \beta^c}$,
- $\Psi_N = \prod_{j=1}^N \varphi(x_j,t) \otimes |0\rangle_{\mathcal{F}} \wedge \alpha = 0 \Rightarrow \beta^b = \beta^c = 0$.

$$\beta^d[\Psi_N, \varphi, \alpha] := \langle\!\langle \Psi_N, \left(\frac{H_N^{PF}}{N} - \mathcal{E}^{HM}[\varphi, \alpha] \right)^2 \Psi_N \rangle\!\rangle,$$

where

$$\mathcal{E}^{HM}[\varphi, \alpha] := \langle \varphi, (-i\nabla - \boldsymbol{A}_\kappa(t))^2 \varphi \rangle + \frac{1}{2} \langle \varphi, \left(v \star |\varphi|^2 \right) \varphi \rangle + \frac{1}{2} \int d^3y \; \boldsymbol{E}_\kappa^2(y,t) + \left(\nabla \times \hat{\boldsymbol{A}}_\kappa \right)^2 (y,t).$$

β^d restricts our consideration to Many-body states, whose energy per particle only fluctuates little around the energy of the effective system:

- $d_t \beta^d = 0$,
- $\Psi_N = \prod_{j=1}^N \varphi(x_j, t) \otimes |0\rangle_{\mathcal{F}} \wedge \alpha = 0 \Rightarrow \beta^d \leq \frac{C}{N}$.

$$\beta^d[\Psi_N, \varphi, \alpha] := \langle\!\langle \Psi_N, \left(\frac{H_N^{PF}}{N} - \mathcal{E}^{HM}[\varphi, \alpha] \right)^2 \Psi_N \rangle\!\rangle,$$

where

$$\mathcal{E}^{HM}[\varphi, \alpha] := \langle \varphi, (-i\nabla - \boldsymbol{A}_\kappa(t))^2 \varphi \rangle + \frac{1}{2} \langle \varphi, \left(v \star |\varphi|^2 \right) \varphi \rangle + \frac{1}{2} \int d^3y \ \boldsymbol{E}_\kappa^2(y,t) + \left(\nabla \times \hat{\boldsymbol{A}}_\kappa \right)^2 (y,t).$$

β^d restricts our consideration to Many-body states, whose energy per particle only fluctuates little around the energy of the effective system:

- $d_t \beta^d = 0$,
- $\Psi_N = \prod_{j=1}^N \varphi(x_j, t) \otimes |0\rangle_{\mathcal{F}} \wedge \alpha = 0 \Rightarrow \beta^d \leq \frac{C}{N}$.

Conclusions and Outlook

Remarks:

- unpublished but soon on the arXiv,
- method can be used to derive the Schrödinger-Klein-Gordon system from the Nelson model,
- UV-cutoff is essential, but can be chosen N-dependent.

Outlook:

- Nelson model with electrons,
- Renormalized Nelson model,
- model for gravitons.

Conclusions and Outlook

Remarks:

- unpublished but soon on the arXiv,
- method can be used to derive the Schrödinger-Klein-Gordon system from the Nelson model,
- UV-cutoff is essential, but can be chosen N-dependent.

Outlook:

- Nelson model with electrons,
- Renormalized Nelson model,
- model for gravitons.

Conclusions and Outlook

Remarks:

- unpublished but soon on the arXiv,
- method can be used to derive the Schrödinger-Klein-Gordon system from the Nelson model,
- UV-cutoff is essential, but can be chosen N-dependent.

Outlook:

- Nelson model with electrons,
- Renormalized Nelson model,
- model for gravitons.

Conclusions and Outlook

Remarks:

- unpublished but soon on the arXiv,
- method can be used to derive the Schrödinger-Klein-Gordon system from the Nelson model,
- UV-cutoff is essential, but can be chosen N-dependent.

Outlook:

- Nelson model with electrons,
- Renormalized Nelson model,
- model for gravitons.

Conclusions and Outlook

Remarks:

- unpublished but soon on the arXiv,
- method can be used to derive the Schrödinger-Klein-Gordon system from the Nelson model,
- UV-cutoff is essential, but can be chosen N-dependent.

Outlook:

- Nelson model with electrons,
- Renormalized Nelson model,
- model for gravitons.

Remarks:

- unpublished but soon on the arXiv,
- method can be used to derive the Schrödinger-Klein-Gordon system from the Nelson model,
- UV-cutoff is essential, but can be chosen N-dependent.

Outlook:

- Nelson model with electrons,
- Renormalized Nelson model,
- model for gravitons.

Thank you for listening!