# Non-linear facets of the electromagnetic quantum field

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# Outline

1. The linear e.m. quantum field
2. The universal $\mathrm{C}^*$-algebra
3. Nonlinearity, topological charges and quantum currents

## Plan

- We review basic ideas leading to the universal $C*$-algebras of the e.m. quantum field, and recall key properties.
- We show how non-linear deformations of the e.m. field leads to some interesting representations of the universal $C^*$-algebra in which non linearity is encoded in a violation of a form of regularity. In particular we construct representations with quantum currents and representations with topological charges.

The talk is based on a joint work with D.Buchholz, F.Ciolli and E.Vasselli [LMP 16] and on a work in progress with the same authors.

## n-Forms on Minkowski spacetime

- *Minkowski spacetime*: $\mathbb{R}^4$ with signature $(+,-,-,-)$. $\perp$ *spacelike separation*.
- $\mathcal{D}_k$ set smooth $k$-forms with compact support in the Minkowski spacetime. $f, h$ are spacelike separated, $f \perp h$, whenever

$$\mathrm{supp}(f) \perp \mathrm{supp}(h) \ .$$

- $d : \mathcal{D}_k \to \mathcal{D}_{k+1}$ , $d^2 = 0$ *differential operator*
- $\star : \mathcal{D}_k \to \mathcal{D}_{4-k}$ , $\star\star = (-)^{k+1}\, id_k$ *Hodge dual*
- $\delta : \mathcal{D}_{k+1} \to \mathcal{D}_k$ , $\delta := -\star d \star$ *co-differential (gen. divergence)*

$$\delta^2 = 0 \quad , \quad \Box = \delta d + d\delta$$

- $\mathcal{C}_k$ set of co-closed $k$-forms (divergence-free): $\delta f = 0$.

- **Geometrical examples**: $f \in \mathcal{D}_0$ a test function with $0 \in \mathrm{supp}(f)$ and a singular $k$-simplex $\chi : [0,1]^k \to \mathbb{R}^4$, let $f_\chi$ be the $k$-form

$$f_\chi(x) := \int f(x-\chi)d\chi$$

then $\mathrm{supp}(f_\chi) \subseteq \mathrm{supp}(f) + \chi$ and the Stokes theorem reads

$$\delta f_\chi = f_{\partial\chi}$$

We call these forms smearing chains. Note that if $\chi$ is a cycle i.e. $\partial\chi = 0$, then $\delta f_\chi = 0$. We shall refer in this case as smearing cycles or diveregece-free forms.

# Outline

# The e.m. quantum field and the intrinsic vector potential

The e.m. quantum field $F$ linear mapping $F : \mathcal{D}_2 \ni h \to F(h) \in \mathscr{A}$ to some $*$-algebra $\mathscr{A}$ s.t.

(i) Causality

$$h_1 \perp h_2 \quad \Rightarrow \quad [F(h_1), F(h_2)] = 0 \ ,$$

(ii) $1^{st}$ Maxwell equation

$$dF(\tau) := F(\delta\tau) = 0 \ , \qquad \tau \in \mathcal{D}_3 \ .$$

From these the $2^{nd}$ Maxwell equation

$$j(f) := F(df) = \delta F(f) \ , \qquad f \in \mathcal{D}_1$$

where $j(f)$ is the conserved current: $\delta j = 0$.

Observation: the intrinsic vector potential satisfies

$$F(h) = dA(h) = A(\delta h)$$

and $\delta h$ is a divergence-free 1-form $\mathcal{C}_1$. Local Poincaré lemma:

$$f \in \mathcal{C}_1 \quad \Rightarrow \quad \exists \widehat{f} \in \mathcal{D}_2 \quad , \quad \delta \widehat{f} = f \ .$$

The intrinsic vector potential is defined as

$$A(f) := F(\widehat{f}) \ , \qquad f \in \mathcal{C}_1 \ .$$

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- By local Poincaré Lemma + $1^{st}$ Maxwell Eq., $A(f)$ independent of the choice of the co-primitive $\widehat{f}$.
- The vector potential induces a (strong) causality $f_1, f_2 \in \mathcal{C}_1$

$$\boxed{f_1 \times f_2 \quad , \quad \exists \widehat{f_1}, \widehat{f_2} \in \mathcal{D}_2 \ , \ \delta \widehat{f_1} = f_1 \ , \ \delta \widehat{f_2} = f_2 \quad , \quad \widehat{f_1} \perp \widehat{f_2}}$$

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- The $1^{st}$ Maxwell equation $dF = d^2 A = 0$
- The conserved current: $j = \delta F = \delta dA$.
- Covariance: $\alpha_P : \mathcal{C}_1 \to \mathcal{C}_1$ with $(\alpha_P f)^\mu := (Pf)^\mu \circ P^{-1}$ then

$$\alpha_P \circ A = A \circ \alpha_P \ , \qquad P \in \mathcal{P}_+^\uparrow \ .$$

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Basic question: understand strong causality

$$f_1 \perp f_2 \ \underset{\overset{?}{\Longrightarrow}}{\Longleftarrow} \ f_1 \times f_2$$

## Stronger (easier to hadle) condition

(i) for any $f \in \mathcal{C}_1$ and any open $U \supset \mathrm{supp}(f)$ there exists a coprimitive $\widehat{f}$ of $f$ with $\mathrm{supp}(\widehat{f}) \subset U$.

(ii) $\perp \;\Rightarrow\; \times$

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Then $(i) \;\Rightarrow (ii)$.

Counterexample to (i): $g \in \mathcal{D}_0$ with $\int g \neq 0$ and a closed curve $\gamma$

$$g^{\mu}_{\gamma}(x) := \int_0^1 g(x-\gamma(t))\,\dot{\gamma}(t) \quad , \quad g^{\mu}_{\gamma} \in \mathcal{C}_1$$

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Cohomological obstruction:

$$\boxed{H^2(\mathbb{R}^4 \setminus \mathrm{supp}(f)) \overset{de\ Rham}{\cong} H_2(\mathbb{R}^4 \setminus \mathrm{supp}(f)) \overset{Alexander\ Duality}{\cong} H^1(\mathrm{supp}(f))\ .}$$

So what can be said about the above relation ?

# Causal Poincaré Lemma and centrality

Causal Poincaré Lemma: given a double cone $\mathcal{O}$ and $f \in \mathcal{C}_1$ with $\mathrm{supp}(f) \perp \mathcal{O}$, there is $\widehat{f} \in \mathcal{D}_2$ with $\delta\widehat{f} = f$ and $\mathrm{supp}(\widehat{f}) \perp \mathcal{O}$.

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Strongest invariance result

Cohomological invariance: if $f_1 \perp f_2$ then $[A(f_1), A(f_2)]$ is independent of co-cohomology class of $f_1$ w.r.t.the causal complement of $\mathrm{supp}(f_2)$ i.e.

$$h \in \mathcal{D}_2,\ \delta h = f_1 - f\ , \mathrm{supp}(h) \perp \mathrm{supp}(f_2) \quad \Rightarrow \quad [A(f_1), A(f_2)] = [A(f), A(f_2)]$$

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- Translation invariance

$$f_1 \perp f_2 \quad \Rightarrow \quad [A(f_{1,x}), A(f_{2,x})] = [A(f_1), A(f_2)]\ , \qquad \forall x \in \mathbb{R}^4$$

- Dilation invariance

$$f_1 \perp f_2 \quad \Rightarrow \quad [A(\tau_\lambda(f_1)), A(\tau_\lambda(f_2))] = \lambda^{-6}\,[A(f_1), A(f_2)]\ , \qquad \forall \lambda > 0$$

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- Centrality (topological charges ?) by translation invariance

$$\boxed{f_1 \perp f_2 \quad \Rightarrow \quad [[A(f_1), A(f_2)]\ ,\ A(f)] = 0\ , \qquad \forall f \in \mathcal{C}_1}$$

# Outline

# The universal C*-algebra of the e.m. quantum field

Let $\mathcal{U}$ be the group generated by $U : \mathbb{R} \times \mathcal{C}_1 \ni (a, f) \to U(a, f)$ s.t.

(i) $U(a,f)^* = U(-a,f) \quad , \quad U(0,f) = 1 \quad , \quad U(a,f)\,U(b,f) = U(a+b,f)$;

(ii) $f_1 \times f_2 \quad \Rightarrow \quad U(a_1,f_1)\,U(a_2,f_2) = U(1, a_1 f_1 + a_2 f_2)$;

(iii) $f_1 \perp f_2 \quad \Rightarrow \quad \lfloor U(a,f), \lfloor U(a_1,f_1), U(a_2,f_2) \rfloor \rfloor = 1$

where $\lfloor , \rfloor$ is the group commutator. The Poincaré group acts on $\mathcal{U}$: $P(a,f) := (a, Pf)$ for any $P \in \mathcal{P}_+^\uparrow$. The universal C*-algebra of the e.m. field $\mathfrak{U}$ is the full group $\mathrm{C}^*$-algebra of $\mathcal{U}$.

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The correspondence $\mathcal{O} \mapsto \mathfrak{U}(\mathcal{O})$ where $\mathcal{O}$ double cone and

$$\mathfrak{U}(\mathcal{O}) := \mathrm{C}^*\{U(a, f) \ , \ \mathrm{supp}(f) \subset \mathcal{O}\} \subset \mathfrak{U}$$

is a net of $C^*$-algebras which satisfying the Haag-Kastler axioms

- $\mathcal{O}_1 \subset \mathcal{O}_2 \quad \Rightarrow \quad \mathfrak{U}(\mathcal{O}_1) \subseteq \mathfrak{U}(\mathcal{O}_2)$ (Isotony)
- $\mathcal{O}_1 \perp \mathcal{O}_2 \quad \Rightarrow \quad [\mathfrak{U}(\mathcal{O}_1), \mathfrak{U}(\mathcal{O}_2)] = 0$ (Causality)
- $\alpha_P(\mathfrak{U}(\mathcal{O})) = \mathfrak{U}(P\mathcal{O}) \ , \ P \in \mathcal{P}_+^\uparrow$ (Covariance)

excepts *Primitivity*: there is a non-trivial center.

## Meaningful states and representations

A state $\omega$ of the algebra $\mathfrak{U}$ is

- regular (strongly regular) if
$$a_1, \ldots, a_n \mapsto \omega(U(a_1, g_1) \cdots U(a_n, g_n))$$
are continuous (smoooth with tempered derivatives at 0)
- verifies condition L if it is strongly regular and
$$\frac{d}{da}\omega(V\, U(a, g_1)\, U(a, g_2)\, U(a, -g_1 - g_2)\, W) \restriction_{a=0} = 0$$

Note that

- $\omega$ strongly regular, $(\Omega, \pi, \mathcal{H})$ GNS of $\omega$, There exist selfadjoint operators $A_\pi(f)$ with common stable core $\mathcal{D} \subseteq \mathcal{H}$ such that $\pi(U(a, f)) = e^{iaA_\pi(f)}$.
- property $L$, $a_1 A_\pi(f_1) + a_2 A_\pi(f_2) = A_\pi(a_1 f_1 + a_2 f_2)$ on $\mathcal{D}$

Vacuum state $\omega$: pure Poincaré invariant state of $\mathfrak{U}$ s.t.

- $\mathcal{P}_+^\uparrow \ni P \to \omega(A\alpha_P(B))$ continuous ;
- $\mathbb{R}^4 \ni p \to \int e^{ipx}\omega(A\alpha_x(B))\, d^4x \in \overline{V}_+$ (spectral codition)

Consequences:

(i) Any vacuum state $\omega$ is determined by the generating functional

$$f \mapsto \omega(U(1,f)) \ , \quad f \in \mathcal{C}_1 \ ,$$

(analyticity and EOW theorem)

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Applications: $\omega$ vacuum state satisfying the property $L$.

- Zero current, $j = 0$ (recall $j_\pi(f) = A_\pi(\delta df)$), then

$$\omega_0(U(1,f)) = e^{-c\langle f,f\rangle} \ , \qquad f \in \mathcal{C}_1$$

the free electromagnetic field in Fock representation with $c > 0$

- Classical current (central current), then

$$\omega(U(1,f)) = e^{ij_\pi(G_0(f))}\omega_0(U(1,f)$$

where $G_0$ Green's function of $\Box$.

# Outline

# Topological charges

Aim: we have seen that

$$f_1 \perp f_2 \;\Rightarrow\; \lfloor U(1,f_1), U(1,f_2) \rfloor \text{ is central}$$

we want to construct representations of $\mathfrak{U}$ where these central elements are not trivial.

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CONJECTURE: we believe that

$$f_1 \perp f_2 \quad\Rightarrow\quad [A(f_1), A(f_2)] = 0$$

This clearly would imply there does not exists any causal linear symplectic form which is not trivial on pair $f_1, f_2$ with $f_1 \perp f_2$.

# Results supporting the conjecture

The conjecture is verified in a vacuum representation (by K-L representation).

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General case: partial results from Cohomological invariance.

1. Consider a smearing cycle $g_\gamma$. i.e. $g \in \mathcal{D}_0$, $\gamma$ closed curve, and $g_\gamma(x) = \int_0^1 g(x - \gamma(t))\,\dot\gamma(t)\,dt$. The Poincaré group has a geometrical action on smearing cycles
$$(P, g_\gamma) \mapsto g_{P\gamma}\ ,\ P \in \mathcal{P}_+^\uparrow$$
which, by cohomological invariance, leaves invariant
$$[A(g_{P\gamma}), A(g'_{P\gamma'})] = [A(g_\gamma), A(g'_{\gamma'})]\quad ,\quad g_\gamma \perp g'_{\gamma'}$$
2. $\gamma$ the circumference $(y+1)^2 + z^2 = 4$ and $\beta$ the circumference $x^2 + (y-1)^2 = 4$ there is $P \in \mathcal{P}_+^\uparrow$ s.t.
$$P\gamma = \beta\ ,\ P\beta = \gamma\ .$$
So for $g$ s.t. $g_\gamma \perp g_\beta$ we have
$$[A(g_\gamma), A(g_\beta)] = [A(g_\beta), A(g_\gamma)] \ \Rightarrow\ [A(g_\beta), A(g_\gamma)] = 0$$

**Main result:** Given $f, \tilde{f} \in \mathcal{C}_1$. Let $\mathcal{O}$ be a double cone and $\gamma$ a closed simple curve s.t.

(i) $H_1(\mathcal{O} + \gamma) \cong H_1(\gamma) = \mathbb{Z}$

(ii) $\mathrm{supp}(\tilde{f}) \subset \mathcal{O} + \gamma$

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- This clearly does not prove the conjecture, because of the restriction given by $\mathcal{O} + \gamma$.
- Anyway, the conjecture, if verified, implies that the non-triviality of $[A(f_1), A(f_2)]$ for $f_1 \perp f_2$ is in contradiction with linearity.

# The example from non-linearity

- Given a 2-form $h$ define

$$H^{\mu\nu} := \int h^{\mu\nu}(x)\, d^4x \quad , \quad H^2 := H^{\mu\nu} H_{\mu\nu} \ .$$

$H^2$ in an invariant and we say that $h$ is of Electric type whenever $H^2 > 0$ and of Magnetic type if $H^2 < 0$.

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- Let $F_0$ be the free e.m. quantum field. If $h \in \mathcal{D}_2$ has connected support we define

$$F(h) := \theta_+(H^2)F_0(h) + \theta_-(H^2)F_0(\star h)$$

  and 0 whenever $H^2 = 0$. $\theta_+$ step function and $\theta_- = 1 - \theta_+$.

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- for any $h_1, h_2 \in \mathcal{D}_2$ with connected support

$$[F(h_1), F(h_2)] = \Big(\theta_+(H_1^2)\theta_+(H_2^2) - \theta_-(H_1^2)\theta_-(H_2^2)\Big) \cdot [h_0(h_1), h_0(h_2)] +$$
$$- \Big(\theta_+(H_1^2)\theta_-(H_2^2) - \theta_-(H_1^2)\theta_+(H_2^2)\Big)\, [h_0(h_1), h_0(\star h_2)]$$

- If $h_1 \perp h_2$ then

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- Roberts shown that

$$[F_0(g_{\sigma_1}), F_0(\star g_{\sigma_1})] = c \cdot \mathbb{1} \quad , \quad c \neq 0$$

for smearing chains where $\sigma_1, \sigma_2$ surfaces with spacelike separated linked boundaries $\partial\sigma_1$ and $\partial\sigma_2$ and laying in the subspace $t = 0$. So they are all of magnetic type and $[F(g_{\sigma_1}), F(g_{\sigma_2})] = 0$

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  for smearing chains where $\sigma_1, \sigma_2$ surfaces with spacelike separated linked boundaries $\partial\sigma_1$ and $\partial\sigma_2$ and laying in the subspace $t = 0$. So they are all of magnetic type and $[F(g_{\sigma_1}), F(g_{\sigma_2})] = 0$

- However we can take $h \in \mathcal{D}_2$ of electric type such that $h + g_{\sigma_2}$ is of electric type and $h \perp g_{\sigma_1}$. Then

$$[F(g_{\sigma_1}), F(h + g_{\sigma_2})] = -[F_0(g_{\sigma_1}), F_0(\star h + \star g_{\sigma_2})] = [F_0(g_{\sigma_1}), F_0(\star g_{\sigma_2})] = c\mathbb{1}.$$

  and we have central elements.

- If $h$ has an (infinite) countable connected components $\{h_k\}$ we have

$$h^\sharp = \left( \sum_{k=1}^{\infty} \sharp h_k \right) \in \mathcal{D}_2 \quad , \quad \sharp := \begin{cases} id \, , & H^2 > 0 \\ \star \, , & H^2 < 0 \, . \end{cases}$$

and note that $(h_1 + h_2)^\sharp = h_1^\sharp + h_2^\sharp$ if $h_1, h_2$ have disjoint supports. Setting

$$F(h) := F_0(h^\sharp) \, , \qquad \forall h \in \mathcal{D}_2 -$$

Clearly $F$ is not linear but $F(h_1 + h_2) = F(h_1) + F(h_1)$ if $h_1, h_2$ have disjoint supports.

***Thm.*** Let

$$U(a, f) := \exp(iF(a\widehat{f})) = \exp(iaA(f)) \, , \qquad f \in \mathcal{C}_1 \quad , \delta\widehat{f} = f$$

Then $U(a, f)$ is covariant, homogeneous $U(a, f) = U(1, af)$, and

(i) $U(a, f_1)\, U(b, f_2) = U(1, af_1 + bf_2)$ if whenever $f_1 \times f_2$

(ii) $\lfloor U(1, f_1), U(1, f_2) \rfloor$, with $f_1 \perp f_2$, is central and not trivial.

However *property L is violated*. The local algebras are the same as those generated by the free theory: $U_0(a, f) := \exp(iF_0(a\widehat{f}))$.

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- Let $f \mapsto j(f)$, $f \in \mathcal{D}_1$, be a conserved current: $\delta j(g) = j(dg) = 0$ for any $g \in \mathcal{D}_0$. If $h \in \mathcal{D}_2$ has connected support define

$$\boxed{F(h) := \begin{cases} j(f) \ , & h = df \\ j(\delta h) \ , & otherwise \end{cases}}$$

- Definition independent of the choice of primitive: if $df_1 = h$, the $d(f - f_1) = 0$. Local Poincaré Lemma

$$\exists g \in \mathcal{D}_0 \ , \ dg = f - f_1 \quad \Rightarrow \quad j(f) = j(f_1 - dg) = j(f_1) - \delta j(g) \overset{conservation \ \ law}{=} j(f_1)$$

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- $2^{nd}$ Maxwell equation:

$$\delta F(f) = F(df) = j(f) \qquad f \in \mathcal{D}_1$$

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THANK YOU !!