QCD on an infinite lattice: Dynamics and ground states.

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The lattice.

Start with the following lattice data:

A triple $\Lambda := (\Lambda^0, \Lambda^1, \Lambda^2)$ consisting of

• $\Lambda^0 := \mathbb{Z}^3$ i.e. the unit cubic lattice – its elements are called sites.

• Let $\widetilde{\Lambda}^1$ be the set of all directed edges (or links) between nearest neighbours in Λ^0.
Let $\Lambda^1 \subset \widetilde{\Lambda}^1$ denote a choice of orientation i.e. for each $(x, y) \in \widetilde{\Lambda}^1$, Λ^1 contains either (x, y) or (y, x) but not both.

• Let $\widetilde{\Lambda}^2$ be the set of all directed faces (or plaquettes) of the unit cubes comprising the lattice. Let $\Lambda^2 \subset \widetilde{\Lambda}^2$ be a choice of orientation.

• To consider subsets of the lattice, for a connected subgraph $S \subset (\Lambda^0, \Lambda^1)$, we let Λ^0_S be the set of all vertices in S, $\Lambda^1_S \subset \Lambda^1$ is the set of links which are edges in S, and $\Lambda^2_S \subset \Lambda^2$ is the set of those plaquettes whose sides are all in S.

Classical gauge kinematics.

On Λ define a classical matter field with a gauge connection acting on it:

• The gauge group: fix a connected, compact semisimple Lie group G, and define the local gauge group by $\prod_{x\in\Lambda^0} G = G^{\Lambda^0} = \{\zeta : \Lambda^0 \to G\}$

• The space of internal degrees of freedom of the matter field:
Let $(\mathbf{V}, (\cdot,\cdot)_{\mathbf{V}})$ be a finite dimensional complex Hilbert space on which G acts smoothly as unitaries, so we take $G \subset U(\mathbf{V})$.
The classical matter fields are elements of $\prod_{x\in\Lambda^0} \mathbf{V}$, on which the local gauge group acts by pointwise multiplication.

• The classical gauge connections are maps $\Phi : \Lambda^1 \to G$,
i.e. elements of $\prod_{\ell\in\Lambda^1} G$.
(Wilson approach - justified by taking $\Phi(\ell)$ to be parallel translation along the link ℓ w.r.t. a given connection in $\mathbb{R}^3$).

Classical gauge kinematics.

The full classical configuration space is thus $\big(\prod_{x\in\Lambda^0}\mathbf{V}\big)\times\big(\prod_{\ell\in\Lambda^1}G\big)$,
and the local gauge group $\prod_{x\in\Lambda^0}G$ acts on it by

$$\Big(\prod_{x\in\Lambda^0}v_x\Big)\times\Big(\prod_{\ell\in\Lambda^1}g_\ell\Big)\mapsto\Big(\prod_{x\in\Lambda^0}\zeta(x)\cdot v_x\Big)\times\Big(\prod_{\ell\in\Lambda^1}\zeta(x_\ell)\,g_\ell\,\zeta(y_\ell)^{-1}\Big)$$

where $\ell=(x_\ell,y_\ell)$ and $\zeta\in\prod_{x\in\Lambda^0}G$.
Note that the orientation of links in Λ^1 was used in the action because it treats the x_ℓ and y_ℓ differently.

(The gauge action on the second part comes from a gauge transformation of a parallel translation from x_ℓ to y_ℓ along ℓ).

Quantum GT

QCD on a finite lattice.

For the quantum theory on a finite lattice:

Fix a finite connected subgraph S of the graph (Λ^0, Λ^1) and let $\Lambda_S := (\Lambda_S^0, \Lambda_S^1, \Lambda_S^2)$.
(Simplify notation by omitting the subscript S in this section.)

Given this finite lattice Λ^0, the model quantizes the classical model on Λ^0 above, by

• replacing for each lattice site $x \in \Lambda^0$, the classical matter configuration space $\mathbf{V}$ with the algebra for a fermionic particle on $\mathbf{V}$ (the quarks),

• for each link $\ell \in \Lambda^1$ we replace the classical connection configuration space G by an algebra which describes a bosonic particle on G (the gluons).

QCD on a finite lattice.

The matter field (quarks):

Equip the space of classical matter fields $\prod_{x\in\Lambda^0} \mathbf{V}$ with the natural pointwise inner product $\langle f, h\rangle = \sum_{x\in\Lambda^0} \big(f(x),\, h(x)\big)_{\mathbf{V}}$.

For the quantized matter fields, take the CAR-algebra

$$\mathfrak{F}_\Lambda := \mathrm{CAR}\big(\prod_{x\in\Lambda^0} \mathbf{V}\big).$$

That is, for each classical matter field $f \in \prod_{x\in\Lambda^0} \mathbf{V}$, we associate a fermionic field $a(f) \in \mathfrak{F}_\Lambda$, and these satisfy the usual CAR–relations:

$$\{a(f), a(h)^*\} = \langle f, h\rangle \mathbf{1} \quad \text{and} \quad \{a(f), a(h)\} = 0 \quad \text{for} \quad f, h \in \prod_{x\in\Lambda^0} \mathbf{V}$$

where $\{A, B\} := AB + BA$ and $\mathfrak{F}_\Lambda$ is generated by the set of all $a(f)$.

QCD on a finite lattice.

A quantum gauge connection field will be written at each link ℓ in terms of a bosonic particle on the configuration space G.

This is given in a generalized Schrödinger representation on $L^2(G)$ by the set of operators $\{U_g,\ T_f \mid g \in G,\ f \in L^\infty(G)\}$ where:

$$(U_g\varphi)(h) := \varphi(g^{-1}h) \quad \text{and} \quad \big(T_f\varphi)(h) := f(h)\varphi(h) \quad \text{for} \quad \varphi \in L^2(G), \tag{1}$$

$g,\ h \in G$ and $f \in L^\infty(G)$, and it is irreducible in the sense that the commutant of $U_G \cup T_{L^\infty(G)}$ consists of the scalars.

There is a natural cyclic invariant unit vector $\psi_0 \in L^2(G)$ given by the constant function $\psi_0(h) = 1$ for all $h \in G$ (assuming that the Haar measure of G is normalized). Then $U_g\psi_0 = \psi_0$, and ψ_0 is cyclic.

QCD on a finite lattice.

We have the intertwining relation $U_g T_f U_g^* = T_{\lambda_g(f)}$ where

$$\lambda : G \to \mathrm{Aut}\, C(G)\,, \quad \lambda_g(f)(h) := f(g^{-1}h) \quad \text{for} \quad g,\, h \in G \qquad (2)$$

is the usual left translation.

At this point we can construct the kinematics algebra.

As (U, T) is a covariant representation of the left translation action λ,

it is natural to take for our kinematics algebra on a link ℓ the crossed product C*-algebra $C(G) \rtimes_\lambda G$ whose representations are exactly the covariant representations of the C^*-dynamical system defined by λ.

The algebra $C(G) \rtimes_\lambda G$ is called the generalised Weyl algebra, and $C(G) \rtimes_\lambda G \cong \mathcal{K}\big(L^2(G)\big)$.

If $\pi_0 : C(G) \rtimes_\lambda G \to \mathcal{B}(L^2(G))$ is the generalized Schrödinger representation, then $\quad \pi_0\big(C(G) \rtimes_\lambda G\big) = \mathcal{K}\big(L^2(G)\big)$.

QCD on a finite lattice.

(Von Neumann uniqueness)
Since $\mathcal{K}\big(L^2(G)\big)$ has only one irreducible representation up to unitary equivalence, it follows that the generalized Schrödinger representation is the unique irreducible covariant representation of λ (up to equivalence).

The operators U_g and T_f are not compact, so they are not in $\mathcal{K}\big(L^2(G)\big) = \pi_0\big(C(G) \rtimes_\lambda G\big)$, but are in fact in its multiplier algebra.

If one chose $C^*(U_G \cup T_{L^\infty(G)})$ as the kinematics algebra instead of $C(G) \rtimes_\lambda G$, then this would have many inappropriate representations, e.g. covariant representations for $\lambda : G \to \mathrm{Aut}\, C(G)$ where the implementing unitaries are discontinuous w.r.t. G.

QCD on a finite lattice.

We combine these C*-algebras into the kinematics field algebra, which is

$$\mathfrak{A}_\Lambda := \mathfrak{F}_\Lambda \otimes \bigotimes_{\ell \in \Lambda^1} \big(C(G) \rtimes_\lambda G \big)$$

which is is well–defined as Λ^1 is finite, and the cross–norms are unique as all algebras in the entries are nuclear.

(If Λ^1 is infinite, the tensor product is undefined)

Since $C(G) \rtimes_\lambda G \cong \mathcal{K}\big(L^2(G)\big)$ and $\mathcal{K}(\mathcal{H}_1) \otimes \mathcal{K}(\mathcal{H}_2) \cong \mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, it follows that

$$\bigotimes_{\ell \in \Lambda^1} \big(C(G) \rtimes_\lambda G \big) \cong \mathcal{K}\big(\bigotimes_{\ell \in \Lambda^1} L^2(G) \big) \cong \mathcal{K}(\mathcal{L})$$

as Λ^1 is finite, where $\mathcal{L}$ is a generic infinite dimensional separable Hilbert space. So as $\mathfrak{F}_\Lambda$ is a full matrix algebra,

$$\mathfrak{A}_\Lambda = \mathfrak{F}_\Lambda \otimes \bigotimes_{\ell \in \Lambda^1} \big(C(G) \rtimes_\lambda G \big) \cong \mathfrak{F}_\Lambda \otimes \mathcal{K}\big(\bigotimes_{\ell \in \Lambda^1} L^2(G) \big) \cong \mathcal{K}(\mathcal{L})$$

QCD on a finite lattice.

Thus for a finite lattice there will be only one irreducible representation, up to unitary equivalence. Also, $\mathfrak{A}_\Lambda$ is simple, so all representations are faithful.

The algebra $\mathfrak{A}_\Lambda$ is faithfully and irreducibly represented on

$$\mathcal{H} = \mathcal{H}_F \otimes \bigotimes_{\ell \in \Lambda^1} L^2(G) \qquad \text{by} \qquad \pi = \pi_F \otimes \Big(\bigotimes_{\ell \in \Lambda^1} \pi_\ell \Big)$$

where $\pi_F : \mathfrak{F}_\Lambda \to \mathcal{B}(\mathcal{H}_F)$ is the Fock representation, and $\pi_\ell : C(G) \rtimes_\lambda G \to \mathcal{B}(L^2(G))$ is the generalized Schrödinger representation for the $\ell^{\rm th}$ entry.

QCD on a finite lattice.

Next we identify the explicit operators corresponding to the quantum connection fields and Hamiltonian.
Consider a single link ℓ and the generalized Schrödinger representation π_ℓ.
Given $X \in \mathfrak{g}$, define its associated momentum operator

$$P_X : C^\infty(G) \to C^\infty(G) \quad \text{by} \quad P_X\varphi := i\frac{d}{dt}U(e^{tX})\varphi\Big|_{t=0}.$$

As $P_X = dU(X)$, it defines a representation of the Lie algebra $\mathfrak{g}$, and clearly $P_X\psi_0 = 0$ if $X \neq 0$.

The intertwining relation produces the generalized CCRs:

$$\big[P_X,\ T_f\big]\varphi = iT_{X^R(f)}\varphi \qquad \text{for} \quad f,\varphi \in C^\infty(G),$$

where $X^R \in \mathfrak{X}(G)$ is the associated right-invariant vector field.

QCD on a finite lattice.

In the rep. π_ℓ, we want to identify the quantum connection $\Phi(\ell)$ at ℓ, as well as the G-electrical fields $E_{ij}(\ell)$.

Pick out the component functions of the matrix group G, i.e. define

$$\Phi_{ij}(\ell)(g) := (e_i, g e_j), \quad g \in G,$$

where $\{e_i \mid i = 1, \ldots, k\}$ is an orthonormal basis of $\mathbb{C}^k$. (Using the action of the structure group G on $\mathbb{C}^k$).

The matrix components of the quantum connection are taken to be the operators $T_{\Phi_{ij}(\ell)}$, which transform correctly w.r.t. gauge transformations.

As the $\Phi_{ij}(\ell)$ are matrix components of elements of G, there are obvious relations between them which reflect the structure of G. The C*-algebra generated by the operators $\{T_{\Phi_{ij}(\ell)} \mid i, j = 1, \ldots, k\}$ is $T_{C(G)}$.

QCD on a finite lattice.

Next, we identify the G-electrical fields $E_{ij}(\ell)$.

Choose a basis $\{Y_r \mid r = 1, \ldots, \dim(\mathfrak{g})\} \subset \mathfrak{g}$, with matrix components $(Y_r)_{ij} = (e_i, Y_r e_j)$. Choose the basis to be orthonormal w.r.t. the Killing form, hence $\mathrm{Tr}(Y_r Y_s) = \delta_{rs}$. We then define

$$E_r(\ell) := P_{Y_r}, \qquad E_{ij}(\ell) := \sum_r (Y_r)_{ij} E_r(\ell)$$

Of particular importance for the dynamics, is the operator $E_{ij}(\ell)E_{ji}(\ell)$ (summation convention). We have

$$E_{ij}(\ell)E_{ji}(\ell) = \sum_r P_{Y_r}^2$$

i.e. it is the Laplacian for the left regular representation $U : G \to \mathcal{U}(L^2(G))$ which therefore commutes with all U_g.

QCD on a finite lattice.

The full set of operators which comprises the set of dynamical variables of the model is as follows.
The representation Hilbert space is

$$\mathcal{H} = \mathcal{H}_F \otimes \bigotimes_{\ell \in \Lambda^1} L^2(G) \qquad \text{where} \qquad \pi_F : \mathfrak{F}_\Lambda \to \mathcal{B}(\mathcal{H}_F)$$

the Fock representation of $\mathfrak{F}_\Lambda$.

Then $\pi_F \otimes \mathbf{1} : \mathfrak{F}_\Lambda \to \mathcal{B}(\mathcal{H})$ will be the action of $\mathfrak{F}_\Lambda$ on $\mathcal{H}$.

The quantum connection is given by the set of operators

$$\{ \widehat{T}^{(\ell)}_{\Phi_{ij}(\ell)} \quad | \quad \ell \in \Lambda^1,\ i,j = 1, \ldots, k \} \quad \text{where}$$

$$\widehat{T}^{(\ell)}_f \;:=\; \mathbf{1} \otimes \big(\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes T^{(\ell)}_f \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\big)$$

and $T^{(\ell)}_f$ is the multiplication operator on the ℓ^{th} factor,
hence $\widehat{T}^{(\ell)}_{\Phi_{ij}(\ell)}$ acts as the identity on all the other factors of $\mathcal{H}$.

QCD on a finite lattice.

The quantum G–electrical field $\widehat{E}_r$ is a map from Λ^1 to operators on the dense domain $\mathcal{H}_F \otimes \bigotimes_{\ell\in\Lambda^1} C^\infty(G) \subset \mathcal{H}$, given by

$$\widehat{E}_r(\ell) := \mathbf{1} \otimes \big(\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes P^{(\ell)}_{Y_r} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\big), \quad \ell \in \Lambda^1$$

where $P^{(\ell)}_X$ is the P_X operator on the subspace $C^\infty(G) \subset L^2(G)$ of the $\ell^{\rm th}$ factor.

Obviously these are unbounded, so for the associated C*-algebra we need to look at the unitary groups generated by them.

Next, we construct the Hamiltonian, and define the gauge transformations.

QCD on a finite lattice.

The Hamiltonian contains Wilson loops of plaquettes.

Choose an oriented loop $L = \{\ell_1, \ell_2, \ldots, \ell_m\} \subset \widetilde{\Lambda}^1$, $\ell_j = (x_j, y_j)$, such that $y_j = x_{j+1}$ for $j = 1, \ldots, m-1$ and $y_m = x_1$.

Let $G_k = G$ be the configuration space of ℓ_k. Denote the components of the gauge potential Φ by $\Phi_{ij}(\ell_k) \in C(G_k)$.

Define $W(L) \in C(G_1) \otimes \cdots \otimes C(G_m)$ by:

$$\begin{aligned} W(L)(g_1, \ldots, g_m) &:= \Phi_{i_1 i_2}(\ell_1)(g_1)\, \Phi_{i_2 i_3}(\ell_3)(g_2) \cdots \Phi_{i_{m-1} i_1}(\ell_m)(g_m) \\ &= (e_{i_1}, g_1 g_2 \cdots g_m e_{i_1}) = \mathrm{Tr}(g_1 g_2 \cdots g_m)\,. \end{aligned}$$

(summing over repeated indices). Note that to perform the product we need to fix identifications of G_i with G.

This acts on $\mathcal{H}$ by multiplication operator on the $L^2(G)$–factors corresponding to $\ell_1, \ldots, \ell_m$.

For spinor indices, let $\mathbf{V} = \mathbf{W} \otimes \mathbb{C}^k$, fix an orthonormal basis $\{w_1, \ldots, w_4\}$ of $\mathbf{W}$ and an orthonormal basis $\{e_1, \ldots, e_k\}$ of $\mathbb{C}^k$, to get indices

$$a(w_j \otimes e_n \cdot \delta_x) =: \psi_{jn}(x)$$

for the quark field generators, where the subscript j is the spinor index on which the γ-matrices act, and the n is the gauge index.
Then the Hamiltonian is

$$\begin{aligned} H &= \tfrac{a}{2} \sum_{\ell \in \Lambda^1} E_{ij}(\ell) E_{ji}(\ell) + \tfrac{1}{2g^2 a} \sum_{p \in \Lambda^2} \left(W(p) + W(p)^*\right) \\ &+ i\tfrac{a}{2} \sum_{\ell \in \Lambda^1} \bar{\psi}_{jn}(x_\ell) \left[\underline{\gamma} \cdot (y_\ell - x_\ell)\right]_{ji} \Phi_{nm}(\ell) \psi_{im}(y_\ell) + h.c. \\ &+ ma^3 \sum_{x \in \Lambda^0} \bar{\psi}_{jn}(x) \psi_{jn}(x) \,, \end{aligned} \qquad (3)$$

where a is the assumed lattice spacing; $W(p)$ is the Wilson loop operator for the plaquette $p \in \Lambda^2$; and the vector $y_\ell - x_\ell$ for a link $\ell = (x_\ell, y_\ell)$ is the vector of length a pointing from x_ℓ to y_ℓ. As usual for spinors, $\bar{\psi}_{jn}(x) = \psi_{in}(x)^* (\gamma_0)_{ij}$.

QCD on a finite lattice.

For gauge transformations, recall the action of the local gauge group $\mathrm{Gau}\,\Lambda = \prod_{x\in\Lambda^0} G = \{\zeta : \Lambda^0 \to G\}$ on the classical configuration space.

Classical GT

For the Fermion algebra we define an action $\alpha^1 : \mathrm{Gau}\,\Lambda \to \mathrm{Aut}\,\mathfrak{F}_\Lambda$ by

$$\alpha^1_\zeta(a(f)) := a(\zeta\cdot f) \qquad \text{where} \qquad (\zeta\cdot f)(x) := \zeta(x)f(x) \quad \text{for all} \quad x\in\Lambda^0,$$

and $f \in \prod_{x\in\Lambda^0} \mathbf{V}$ since $f \mapsto \zeta\cdot f$ defines a unitary on $\prod_{x\in\Lambda^0} \mathbf{V}$ where $\zeta \in \mathrm{Gau}\,\Lambda$.

As $\pi_F : \mathfrak{F}_\Lambda \to \mathcal{B}(\mathcal{H}_F)$ is covariant w.r.t. α^1, i.e. there is a (continuous) unitary representation $U^F : \mathrm{Gau}\,\Lambda \to \mathcal{U}(\mathcal{H}_F)$ such that

$$\pi_F(\alpha^1_\zeta(A)) = U^F_\zeta \pi_F(A) U^F_{\zeta^{-1}} \quad \text{for} \quad A \in \mathfrak{F}_\Lambda.$$

QCD on a finite lattice.

If the configuration space G corresponds to a link $\ell = (x_\ell, y_\ell)$, then the gauge transformation is $\zeta \cdot g = \zeta(x_\ell)\, g\, \zeta(y_\ell)^{-1}$ for all $g \in G$.
Define a unitary $W_\zeta : L^2(G) \to L^2(G)$ by

$$(W_\zeta \varphi)(h) := \varphi(\zeta^{-1} \cdot h) = \varphi(\zeta(x_\ell)^{-1}\, h\, \zeta(y_\ell))$$

using the fact that G is unimodular, where the inverse was introduced to ensure that $\zeta \to W_\zeta$ is a homomorphism. Note that $W_\zeta \psi_0 = \psi_0$.

Thus for the full system we define on $\mathcal{H} = \mathcal{H}_F \otimes \bigotimes_{\ell \in \Lambda^1} L^2(G)$ the unitaries

$$\widehat{W_\zeta} := U_\zeta^F \otimes \Big(\bigotimes_{\ell \in \Lambda^1} W_\zeta^{(\ell)} \Big), \quad \zeta \in \mathrm{Gau}\,\Lambda$$

where $W_\zeta^{(\ell)}$ is the W_ζ operator on the ℓ^{th} factor.
The gauge transformation produced by ζ on the system of operators is given by $\mathrm{Ad}(\widehat{W_\zeta})$. (Then H is gauge invariant). Infinite lattice GTs

The infinite lattice - algebras.

Next, we wish to extend this model to the infinite lattice $\Lambda = (\Lambda^0, \Lambda^1, \Lambda^2)$. Fix a finite connected subgraph S of the graph (Λ^0, Λ^1), then for the finite lattice $\Lambda_S := (\Lambda^0_S, \Lambda^1_S, \Lambda^2_S)$ we have the kinematics field algebra:

$$\mathfrak{A}_S = \mathfrak{F}_S \otimes \bigotimes_{\ell \in \Lambda^1_S} \big(C(G) \rtimes_\lambda G\big) \cong \mathcal{K}(\mathcal{L})$$

(abbreviating notation to $\mathfrak{A}_S := \mathfrak{A}_{\Lambda_S}$ and $\mathfrak{F}_S := \mathfrak{F}_{\Lambda_S}$)

For an infinite lattice, we cannot directly extend this definition, as an infinite tensor product of nonunital C*-algebras is undefined. (Adjoining identities will introduce spurious representations)

In CMP **318**, 717–766 (2013) we did construct an infinite tensor product of compact operator algebras, but we could not define the dynamics on it.

Our strategy is as follows.

- We first combine the representations

$$\pi_S = \pi_F \otimes \Big(\bigotimes_{\ell \in \Lambda_S^1} \pi_\ell \Big) \qquad \text{on} \qquad \mathcal{H}_S = \mathcal{H}_F \otimes \bigotimes_{\ell \in \Lambda_S^1} L^2(G)\,.$$

  into one representation,
- then we will define a convenient C*-algebra on it, containing all the local algebras $\mathfrak{A}_S$.
- We prove that a global time evolution can be defined on this algebra.
- For our kinematics algebra, we then take the C*-algebra generated by the orbits of all local algebras.
- The kinematics algebra has nonphysical representations, but we will define the class of "regular" representations for physics, and prove that there are ground states which are regular.

Take an infinite product of generalized Schrödinger representations, one for each $\ell \in \Lambda^1$, w.r.t. the reference sequence $(\psi_0, \psi_0, \ldots)$ where $\psi_0 = 1$ is the constant vector.

Thus the space $\mathcal{H}_\infty$ is the completion of the pre–Hilbert space spanned by finite combinations of elementary tensors of the type

$$\varphi_1 \otimes \cdots \otimes \varphi_k \otimes \psi_0 \otimes \psi_0 \otimes \cdots\,, \quad \varphi_i \in \mathcal{H}_i = L^2(G),\; k \in \mathbb{N}$$

w.r.t. the pre–inner product given by

$$\big(\varphi_1 \otimes \cdots \varphi_k \otimes \psi_0 \otimes \psi_0 \otimes \cdots\,,\; \varphi_1' \otimes \cdots \varphi_k' \otimes \psi_0 \otimes \cdots\big)_\infty := \prod_{i=1}^k (\varphi_i, \varphi_i')\,.$$

Then $\mathcal{L}_S := \bigotimes_{\ell \in \Lambda_S^1} \big(C(G) \rtimes_\lambda G\big)$ acts on $\mathcal{H}_\infty$ as a product representation of π_0 where each factor $\mathcal{L}_\ell := C(G) \rtimes_\lambda G$ acts on the factor of $\mathcal{H}_\infty$ corresponding to ℓ.

Thus all $\mathcal{L}_S$ are faithfully imbedded in $\mathcal{B}(\mathcal{H}_\infty)$, and if S and S' are disjoint, $\mathcal{L}_S$ and $\mathcal{L}_{S'}$ commute.

For the fermions, let

$$\mathfrak{F}_\Lambda := \mathrm{CAR}(\ell^2(\Lambda^0, \mathbf{V}))$$

and note that $\mathfrak{F}_S \subset \mathfrak{F}_\Lambda$ naturally, as the finite dimensional space $\ell^2(\Lambda^0_S, \mathbf{V}) \subset \ell^2(\Lambda^0, \mathbf{V})$.

Now consider the Fock representation $\pi_{\mathrm{Fock}} : \mathfrak{F}_\Lambda \to \mathcal{B}(\mathcal{H}_{\mathrm{Fock}})$ of the CAR–algebra with vacuum vector Ω. The Hilbert space on which we define our infinite lattice model is

$$\mathcal{H} := \mathcal{H}_{\mathrm{Fock}} \otimes \mathcal{H}_\infty.$$

Then by $\mathfrak{F}_S \subset \mathfrak{F}_\Lambda$, we also have a product representation of the local field algebras $\mathfrak{A}_S := \mathfrak{F}_S \otimes \bigotimes_{\ell \in \Lambda^1_S} \mathcal{L}_\ell$ on $\mathcal{H}$. If S and S' are disjoint, then $\mathfrak{A}_S$ and $\mathfrak{A}_{S'}$ will graded–commute w.r.t. the Fermion grading. If we have containment i.e., $R \subset S$, then $\mathfrak{F}_R \subset \mathfrak{F}_S$, but we have $\bigotimes_{\ell \in \Lambda^1_R} \mathcal{L}_\ell \not\subset \bigotimes_{\ell \in \Lambda^1_S} \mathcal{L}_\ell$.

However w.r.t. the natural operator product we have $\mathfrak{A}_R \cdot \mathfrak{A}_S = \mathfrak{A}_S$, hence $\mathfrak{A}_R \subset M(\mathfrak{A}_S)$.

For each finite connected subgraph S of (Λ^0, Λ^1), we define

$$\begin{aligned} \mathcal{B}_S &= \pi_{\text{Fock}}(\mathfrak{F}_S) \otimes \mathcal{B}\Big(\bigotimes_{\ell \in \Lambda^1_S} \mathcal{H}_\ell\Big) \otimes \bigotimes_{\ell \notin \Lambda^1_S} \mathbf{1} \subset \mathcal{B}(\mathcal{H}), \\ \mathcal{H}_S &:= [\mathfrak{A}_S(\Omega \otimes \psi_0^\infty)] = [\mathfrak{F}_S \Omega] \otimes \Big(\bigotimes_{\ell \in \Lambda^1_S} \mathcal{H}_\ell\Big) \otimes \bigotimes_{\ell \notin \Lambda^1_S} \psi_0 \subset \mathcal{H}. \end{aligned}$$

Then $\mathcal{B}_S \subset \mathcal{B}(\mathcal{H})$ restricts to $\mathcal{H}_S \subset \mathcal{H}$, this restriction map is faithful, and has image $\mathcal{B}(\mathcal{H}_S)$. We have containments $\mathcal{B}_S \subseteq \mathcal{B}_T$ if $S \subseteq T$.

We define:

$$\mathcal{A}_{\max} := \varinjlim \mathcal{B}_S = C^*\Big(\bigcup_{S \in \mathcal{S}} \mathcal{B}_S\Big)$$

where in the last inductive limit and union, S ranges over the directed set $\mathcal{S}$ of all finite connected nonempty subgraphs of (Λ^0, Λ^1).
As $\mathcal{A}_{\max}$ contains all the local algebras $\mathfrak{A}_S$, it is a convenient universe in which to construct the time automorphisms.

The infinite lattice - dynamics.

We want to define the dynamics on $\mathcal{A}_{\max} := \varinjlim \mathcal{B}_S$ corresponding to the heuristic Hamiltonian given by

$$\begin{aligned} H &= \frac{a}{2}\sum_{\ell\in\Lambda^1} E_{ij}(\ell)E_{ji}(\ell) + \frac{1}{2g^2a}\sum_{p\in\Lambda^2}(W(p)+W(p)^*) \\ &+ i\frac{a}{2}\sum_{\ell\in\Lambda^1}\bar{\psi}_{jn}(x_\ell)\big[\underline{\gamma}\cdot(y_\ell - x_\ell)\big]_{ji}\Phi_{nm}(\ell)\psi_{im}(y_\ell) + h.c. \\ &+ ma^3\sum_{x\in\Lambda^0}\bar{\psi}_{jn}(x)\psi_{jn}(x)\,, \end{aligned} \tag{4}$$

where now the sums are over an infinite lattice, so are not yet properly defined.

For $S \in \mathcal{S}$, define the local Hamiltonian H_S by summing only over Λ_S:

$$\begin{aligned}
H_S &= H_S^{\rm loc} + H_S^{\rm int} \qquad \text{on} \qquad \mathcal{H}_{\rm Fock} \otimes \mathcal{D}_S \qquad \text{where} \\
H_S^{\rm loc} &:= \tfrac{a}{2} \sum_{\ell \in \Lambda_S^1} E_{ij}(\ell) E_{ji}(\ell) + m a^3 \sum_{x \in \Lambda_S^0} \bar{\psi}_i(x) \psi_i(x) \qquad \text{on} \qquad \mathcal{H}_{\rm Fock} \otimes \mathcal{D}_S \\
H_S^{\rm int} &:= \frac{1}{2g^2 a} \sum_{p \in \Lambda_S^2} \left(W(p) + W(p)^* \right) \\
&\quad + i \tfrac{a}{2} \sum_{\ell \in \Lambda_S^1} \bar{\psi}_{jn}(x_\ell) \left[\underline{\gamma} \cdot (y_\ell - x_\ell) \right]_{ji} \Phi_{nm}(\ell) \psi_{im}(y_\ell) + h.c. \in \mathcal{B}_S
\end{aligned}$$

where $\mathcal{D}_S = \bigotimes_{\ell \in \Lambda_S^1} C^\infty(G) \otimes \bigotimes_{\ell' \notin \Lambda_S^1} L^2(G) \subset \bigotimes_{\ell \in \Lambda^1} L^2(G)$.

For $p = (\ell_1, \ell_2, \ell_3, \ell_4) \in \Lambda^2$, we have

$$\begin{aligned}
W(p) &\in C(G_{\ell_1} \times \cdots G_{\ell_4}) \subset M(\mathcal{L}_S) \subseteq \mathcal{B}_S \qquad \text{if} \qquad p \subset S \\
\text{and} &\qquad \bar{\psi}_{jn}(x_\ell) \left[\underline{\gamma} \cdot (y_\ell - x_\ell) \right]_{ji} \Phi_{nm}(\ell) \psi_{im}(y_\ell) \in \mathfrak{F}_S \otimes C(G_\ell) \subset \mathcal{B}_S
\end{aligned}$$

where $\ell = (x, y) \subset S$.

For each $S \in \mathcal{S}$ we have a local time evolution $\alpha^S : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}_{\max})$ which preserves $\pi(\mathcal{B}_S)$ and acts trivially on $\mathcal{A}_{\max}$ outside of it, given by

$$\alpha_t^S := \mathrm{Ad}(U_S(t)) \quad \text{and} \quad U_S(t) := \exp(itH_S).$$

The local Hamiltonians H_S are strongly affiliated with their local algebras $\mathfrak{A}_S$, i.e. $(i\mathbf{1} - H_S)^{-1} \in \mathfrak{A}_S$.

We can now apply arguments of Nachtergaele and Sims to establish the existence of the full infinite lattice dynamics on $\mathcal{A}_{\max}$.

Theorem (Global dynamics)

For all $A \in \mathcal{A}_{\max}$ and $t \in \mathbb{R}$, the norm limit

$$\lim_{S \nearrow \mathbb{Z}^3} \alpha_t^S(A) =: \alpha_t(A)$$

exists, and defines an automorphism group $t \mapsto \alpha_t \in \mathrm{Aut}(\mathcal{A}_{\max})$. The limit is over increasing sequences in $\mathcal{S}$ such that the union of the graphs in the sequence is the entire connected graph (Λ^0, Λ^1) for the lattice. Furthermore, for each $T > 0$, the limit is uniform w.r.t. $t \in [-T, T]$.

(Sketch proof at end)

The infinite lattice - dynamics.

Having obtained the time evolution $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}_{\max})$, we can now define our kinematics algebra as

$$\mathfrak{A}_\Lambda := C^*\Big(\bigcup_{S\in\mathcal{S}} \alpha_{\mathbb{R}}(\mathfrak{A}_S)\Big) \subset \mathcal{A}_{\max} \subset \mathcal{B}(\mathcal{H}).$$

which is the minimal C*-algebra which contains all the local field algebras, and is preserved by the dynamics.

Theorem

For all $A \in \mathfrak{A}_\Lambda$ we have that $t \mapsto \alpha_t(A)$ is norm continuous, i.e. α is strongly continuous on $\mathfrak{A}_\Lambda \subset \mathcal{A}_{\max}$.

However α is not strongly continuous on $\mathcal{A}_{\max}$.

The infinite lattice - ground states.

Next, we want to show that there are physically reasonable ground states of $\mathfrak{A}_\Lambda$ w.r.t. α.

Definition

A representation π of $\mathfrak{A}_\Lambda$ is **regular** if its restriction to each local algebra $\mathfrak{A}_S$, $S \in \mathcal{S}$, is nondegenerate (i.e. $\pi(\mathfrak{A}_S)$ has no nonzero null spaces). A state is regular if its GNS representation is regular.

If a covariant representation is regular, then it is also nondegenerate on all the time evolved local algebras $\alpha_t(\mathfrak{A}_S)$.

Physical representations should be regular as one requires that there is no S for which all the local observables in $\mathfrak{A}_S$ have zero expectation values w.r.t. some (normalized) vector state in the representation, i.e. the local observables in the kinematics algebra should be visible in any physically realizable state of the system.

The infinite lattice - ground states.

We will construct ground states as weak-* limits of ground states w.r.t. the local time evolutions $\alpha_t^S = \mathrm{Ad}\big(\exp(itH_S)\big) \subset \mathrm{Aut}(\mathcal{A}_{\max})$ (same α^S we used in limit for global time evolution).

We need some spectral information. Separating the bounded and unbounded parts of H_S on $\mathcal{H}_S$ we have:

$$\begin{aligned} H_S &= H_S^{(0)} + H_S^{\mathrm{bound}} \qquad \text{on} \qquad [\mathfrak{F}_S\Omega] \otimes \widetilde{\mathcal{D}}_S \subset \mathcal{H}_S \subset \mathcal{H}_{\mathrm{Fock}} \otimes \mathcal{H}_\infty \\ H_S^{(0)} &:= \frac{a}{2} \sum_{\ell \in \Lambda_S^1} E_{ij}(\ell) E_{ji}(\ell), \qquad H_S^{\mathrm{bound}} \in \mathcal{B}(\mathcal{H}_S) \qquad \text{and} \\ \widetilde{\mathcal{D}}_S &= \bigotimes_{\ell \in \Lambda_S^1} C^\infty(G) \otimes \bigotimes_{\ell' \notin \Lambda_S^1} \psi_0. \end{aligned}$$

Now $H_S^{(0)} = \mathbf{1} \otimes R_S \otimes \mathbf{1}$ where R_S is the group Laplacian for the compact Lie group $G_S := \prod_{\ell \in \Lambda_S^1} G$ on $L^2(G_S) \cong \bigotimes_{\ell \in \Lambda_S^1} L^2(G)$.

The infinite lattice - ground states.

By the theory of elliptic operators on compact Riemannian manifolds, the Laplacian R_S has a spectrum of isolated eigenvalues, and its eigenspaces are finite dimensional, hence it has compact resolvent, i.e. $(i\mathbf{1} - R_S)^{-1} \in \mathcal{K}(L^2(G_S))$.

As $[\mathfrak{F}_S\Omega]$ is finite dimensional, this is also true for $H_S^{(0)}$ on $[\mathfrak{F}_S\Omega] \otimes \widetilde{\mathcal{D}}_S \subset \mathcal{H}_S$, i.e. $\left(i\mathbf{1} - H_S^{(0)}\right)^{-1} \in \mathcal{K}(\mathcal{H}_S)$. Then

$$(i\mathbf{1} - H_S)^{-1} = \left(i\mathbf{1} - H_S^{(0)}\right)^{-1} + (i\mathbf{1} - H_S)^{-1} H_S^{\mathrm{bound}} \left(i\mathbf{1} - H_S^{(0)}\right)^{-1} \in \mathcal{K}(\mathcal{H}_S)$$

hence $H_S = H_S^{(0)} + H_S^{\mathrm{bound}}$ also has discrete spectrum with finite dimensional eigenspaces.

As $H_S^{(0)}$ is positive and unbounded, and H_S^{bound} is bounded, H_S is bounded from below. Thus the lowest point in the spectrum of H_S is an eigenvalue $\lambda_S^{\mathrm{grnd}} \in \mathbb{R}$.

Fix a normalized eigenvector $\Omega_S \in \mathcal{H}_S \subset \mathcal{H}$ associated to the eigenvalue $\lambda_S^{\rm grnd} \in \mathbb{R}$.

Then the vector state $\omega_S(\cdot) := \big(\Omega_S,\, \cdot\,\Omega_S\big)$ is a ground state for the local time evolution $\alpha^S : \mathbb{R} \to {\rm Aut}\big(\mathcal{A}_{\rm max}\big)$ (and for its restriction to $\mathfrak{A}_\Lambda \subset \mathcal{A}_{\rm max}$).

Now fix a strictly increasing sequence $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $S_n \nearrow \mathbb{Z}^3$ as $n \to \infty$. For each $n \in \mathbb{N}$ choose a state ω_n on $\mathcal{B}(\mathcal{H})$ in the norm closed convex hull of vector states

$$A \mapsto \big(\Omega_{S_n},\, A\,\Omega_{S_n}\big), \qquad A \in \mathcal{B}(\mathcal{H}),$$

where $\Omega_{S_n} \in \mathcal{H}_{S_n}$ ranges over the normalized vectors in the lowest eigenspace of H_{S_n}.

This sequence $\{\omega_n\}_{n\in\mathbb{N}}$ need not converge, but by the Banach–Alaoglu theorem, the closed unit ball in $\mathcal{B}(\mathcal{H})^*$ is compact in the weak *–topology, hence the sequence $\{\omega_n\}_{n\in\mathbb{N}}$ has weak *–limit points, and these limit points are states.

From such weak *–limit points we now want to show that we can obtain regular ground states on $\mathfrak{A}_\Lambda$.

The infinite lattice - ground states.

Theorem

For the increasing sequence $S_n \nearrow \mathbb{Z}^3$ we fix S_n to be the lattice cube with corner vertices $(\pm n, \pm n, \pm n)$, which produces the sequence $\{\omega_n\}_{n\in\mathbb{N}}$.
*Let ω_∞ be a weak *–limit point of $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})^*$.*
Then the restriction of ω_∞ to $\mathfrak{A}_\Lambda \subset \mathcal{B}(\mathcal{H})$ is a regular ground state for $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathfrak{A}_\Lambda)$.

There are several sources of nonuniqueness for ground states in this argument. Apart from the possibility of different weak *–limit points of $\{\omega_n\}_{n\in\mathbb{N}}$, there are also different choices of ω_n as the lowest eigenspace of H_{S_n} over which Ω_{S_n} ranges may have dimension higher than one.

Gauge transformations.

We need to define gauge transformations, identify the algebra of physical observables, equip it with the time evolution automorphism group, and prove the existence of a ground state.

The local gauge transformations on the lattice Λ^0 is the group of maps $\gamma : \Lambda^0 \to G$ of finite support, i.e.

$$\begin{aligned} \mathrm{Gau}\,\Lambda &:= G^{(\Lambda^0)} = \big\{\gamma : \Lambda^0 \to G \ \big|\ \big|\mathrm{supp}(\gamma)\big| < \infty\big\}, \\ \mathrm{supp}(\gamma) &:= \{x \in \Lambda^0 \mid \gamma(x) \neq e\}. \end{aligned}$$

It acts on each local field algebra $\mathfrak{A}_S = \mathfrak{F}_S \otimes \bigotimes_{\ell\in\Lambda^1_S} \mathcal{L}_\ell$ by a product action, implemented by a unitary

$$\widehat{W_\zeta} := U^F_\zeta \otimes \Big(\bigotimes_{\ell\in\Lambda^1_S} W^{(\ell)}_\zeta\Big), \quad \zeta \in \mathrm{Gau}\,\Lambda_S,$$

on $\mathcal{H}_F \otimes \bigotimes_{\ell\in\Lambda^1_S} L^2(G)$. Finite lattice GTs

Gauge transformations.

For the full infinite lattice, these unitaries generalize naturally to $\mathcal{H} := \mathcal{H}_{\mathrm{Fock}} \otimes \mathcal{H}_\infty$ by the same formulae, as each $\zeta \in \mathrm{Gau}\,\Lambda$ is of finite support, i.e.

$$\widehat{W}_\zeta := U_\zeta^F \otimes \Big(\bigotimes_{\ell \in \Lambda^1_{\mathrm{supp}'(\zeta)}} W_\zeta^{(\ell)} \Big), \quad \zeta \in \mathrm{Gau}\,\Lambda\,.$$

Here $\mathrm{supp}'(\zeta)$ denotes the subgraph of Λ consisting of all the links which have at least one point in $\mathrm{supp}(\zeta)$.

Hence $\Lambda^1_{\mathrm{supp}'(\gamma)}$ consists of the links which have at least one point in $\mathrm{supp}(\gamma)$. We assumed that $\widehat{W}_\zeta$ acts as the identity on those factors of $\mathcal{H}_\infty$ corresponding to $\ell \notin \mathrm{supp}'(\gamma)$.

This produces a unitary representation $\widehat{W} : \mathrm{Gau}\,\Lambda \to \mathcal{U}(\mathcal{H})$.

Gauge transformations.

The gauge transformation produced by ζ on the system of operators is given by $\mathrm{Ad}(\widehat{W_\zeta})$, and on the local algebras it produces the same gauge transformations as before.

These gauge transformations preserve

$$\mathcal{A}_{\max} = \varinjlim \mathcal{B}_S = C^*\Big(\bigcup_{S\in\mathcal{S}} \mathcal{B}(\mathcal{H}_S)\Big)$$

hence we can use $\mathrm{Ad}(\widehat{W_\zeta})$ to define gauge transformations on our maximal algebra.

The local Hamiltonians H_S are constructed from gauge invariant terms, and hence $\mathrm{Ad}(\widehat{W_\zeta})(e^{itH_S}) = e^{itH_S}$ and from this follows that for all $A \in \mathcal{A}_{\max}$ and $\zeta \in \mathrm{Gau}\,\Lambda$ we have

$$\widehat{W_\zeta}\,\alpha_t(A)\,\widehat{W_\zeta}^* = \alpha_t\big(\widehat{W_\zeta}A\widehat{W_\zeta}^*\big)$$

i.e. the global time evolution also commutes with the gauge transformations.

Gauge transformations.

Thus the gauge transformations $\mathrm{Ad}(\widehat{W_\zeta})$ will preserve all orbits of the global time evolution, and hence $\mathrm{Ad}(\widehat{W_\zeta})$ preserves our kinematics algebra $\mathfrak{A}_\Lambda := C^*\Big(\bigcup_{S\in\mathcal{S}} \alpha_{\mathbb{R}}(\mathfrak{A}_S)\Big)$.

By restriction, the gauge transformations are therefore well-defined on $\mathfrak{A}_\Lambda$, and we will denote the action by $\beta : \mathrm{Gau}\,\Lambda \to \mathrm{Aut}(\mathfrak{A}_\Lambda)$.

We now have:-

Theorem

There is a regular ground state ω *for* $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathfrak{A}_\Lambda)$ *which is gauge invariant, i.e.* $\omega \circ \beta_\zeta = \omega$ *for all* $\zeta \in \mathrm{Gau}\,\Lambda$.

Finally, we would like to identify the physical subalgebra (i.e. enforce the Gauss law constraint).

As the local algebras are copies of the compacts, several different constraint enforcement procedures produced the same result.
E.g. the traditional constraint enforcement method - taking the gauge invariant part of the algebra, then factoring out the residual constraints - produced results coinciding with those of the T–procedure of G. & Hurst.

The concrete constraint reduction can be done in the defining representation of $\mathfrak{A}_\Lambda \subset \mathcal{B}(\mathcal{H})$ as follows.

Using the representation $\widehat{W} : \mathrm{Gau}\,\Lambda \to \mathcal{U}(\mathcal{H})$ of the gauge group which implements the gauge transformations, one defines the gauge invariant subspace

$$\mathcal{H}_G := \{\psi \in \mathcal{H} \mid \widehat{W}_\zeta \psi = \psi \ \ \forall\, \zeta \in \mathrm{Gau}\,\Lambda\}$$

(note that the cyclic vector $\Omega \otimes \psi_0^\infty$ is in $\mathcal{H}_G$).

The algebra of physical observables $\mathcal{P}$ will be the restriction to $\mathcal{H}_G$ of those operators in $\mathfrak{A}_\Lambda$ which preserve $\mathcal{H}_G$, together with their adjoints.

Let P_G be the projection onto $\mathcal{H}_G$, then an $A \in \mathcal{B}(\mathcal{H})$ commutes with P_G iff both A and A^* preserve $\mathcal{H}_G$, hence

$$\{P_G\}' \cap \mathfrak{A}_\Lambda \subseteq P_G \mathcal{B}(\mathcal{H}) P_G + (\mathbf{1} - P_G)\mathcal{B}(\mathcal{H})(\mathbf{1} - P_G).$$

Thus the algebra of physical observables is

$$\mathcal{P} = \left(\{P_G\}' \cap \mathfrak{A}_\Lambda\right) \restriction \mathcal{H}_G$$

i.e. we discard the second part of the decomposition above.

On the local algebras $\mathfrak{A}_S$ this will produce a copy of the algebra of compact operators on the gauge invariant part of $\mathcal{H}_S$.

- The time evolution automorphism group on $\mathfrak{A}_\Lambda$ preserves $\{P_G\}' \cap \mathfrak{A}_\Lambda$, and defines a time evolution automorphism group on the algebra of physical observables $\mathcal{P}$.
- The regular gauge invariant ground state on $\mathfrak{A}_\Lambda$ defines a ground state on $\mathcal{P}$ w.r.t. the time evolution obtained from α.

End slide

Sketch proof.

Theorem (Global dynamics)

For all $A \in \mathcal{A}_{\max}$ and $t \in \mathbb{R}$, the norm limit

$$\lim_{S \nearrow \mathbb{Z}^3} \alpha_t^S(A) =: \alpha_t(A)$$

exists, and defines an automorphism group $t \mapsto \alpha_t \in \mathrm{Aut}(\mathcal{A}_{\max})$. The limit is over increasing sequences in $\mathcal{S}$ such that the union of the graphs in the sequence is the entire connected graph (Λ^0, Λ^1) for the lattice.

(Method of Nachtergaele & Sims)
Fix $T > 0$, a nonempty $R \subset S$ and let $A \in \mathcal{B}_R$. We want to show that the limit of $\alpha_t^S(A)$ as $S \nearrow \mathbb{Z}^3$ exists for $|t| < T$.

Fix a strictly increasing sequence $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $S_n \nearrow \mathbb{Z}^3$ as $n \to \infty$. We need to show that $\{\alpha_t^{S_n}(A)\}_{n\in\mathbb{N}}$ is Cauchy.

$$\text{Notation:} \quad H_S = H_S^{\rm loc} + H_S^{\rm int} \quad \text{where}$$

$$H_S^{\rm loc} := \sum_{\ell\in\Lambda_S^1} H_\ell + \sum_{x\in\Lambda_S^0} H_x \quad \text{where}$$

$$H_\ell := \tfrac{a}{2} E_{ij}(\ell) E_{ji}(\ell) \quad \text{and} \quad H_x := ma^3 \overline{\psi}_i(x)\psi_i(x)$$

$$H_S^{\rm int} := \sum_{p\in\Lambda_S^2} \widetilde{W}(p) + \sum_{\ell\in\Lambda_S^1} B(\ell) =: \sum_{q\in\Lambda_S^i} \Psi(q) \in \mathcal{B}_S$$

$$\text{where} \quad \widetilde{W}(p) := \tfrac{1}{2g^2 a}(W(p) + W(p)^*) \in \mathcal{B}_S$$

$$\text{and} \quad B(\ell) := i\tfrac{a}{2} \overline{\psi}_{jn}(x_\ell)\big[\underline{\gamma}\cdot(y_\ell - x_\ell)\big]_{ji} \Phi_{nm}(\ell)\psi_{im}(y_\ell) + h.c. \in \mathcal{B}_S$$

where $\Lambda_S^i := \Lambda_S^1 \cup \Lambda_S^2$ and $\Psi(p) := \widetilde{W}(p)$ for plaquette p, and $\Psi(\ell) := B(\ell)$ for a link ℓ. Denote $\|\Psi\| := \max\{\|\widetilde{W}\|, \|B\|\}$.

Now for $R \subseteq S_n$:

$$\alpha_t^{S_n}(A) = \tau_t^{S_n}\left(e^{itH_{S_n}^{\rm loc}} A e^{-itH_{S_n}^{\rm loc}}\right) = \tau_t^{S_n}\left(e^{itH_R^{\rm loc}} A e^{-itH_R^{\rm loc}}\right) = \tau_t^{S_n}\big(A(t)\big)$$

where $\tau_t^S := \mathrm{Ad}(e^{itH_S} e^{-itH_S^{\rm loc}})$ and

$$A(t) := e^{itH_R^{\rm loc}} A\, e^{-itH_R^{\rm loc}} \in \mathcal{B}_R\,.$$

Thus it suffices to show that the sequence $\{\tau_t^{S_n}(A)\}_{n\in\mathbb{N}}$ is Cauchy for all $A \in \mathcal{B}_R$. Define

$$U_S(t,s) := e^{itH_S^{\rm loc}} e^{i(s-t)H_S} e^{-isH_S^{\rm loc}}\,.$$

Then by the fundamental theorem of calculus

$$\tau_t^{S_m}(A) - \tau_t^{S_n}(A) = \int_0^t \frac{d}{ds}\Big(U_{S_m}(0,s)U_{S_n}(s,t)A\,U_{S_n}(t,s)U_{S_m}(s,0)\Big)\,ds$$

where the differential and integral is w.r.t. the strong operator topology.

The integrand is for any $\psi \in \mathcal{H}$:

$$\begin{aligned} &\frac{d}{ds}\, U_{S_m}(0,s) U_{S_n}(s,t) A\, U_{S_n}(t,s) U_{S_m}(s,0)\psi \\ &= \quad iU_{S_m}(0,s) e^{isH^{\rm loc}_{S_n}} \Big[N(s), \alpha^{S_n}_{s-t}(A(t)) \Big] e^{-isH^{\rm loc}_{S_n}} U_{S_m}(s,0)\psi \quad \text{where} \end{aligned}$$

$$N(s) \;:=\; \mathrm{Ad}\Big(e^{isH^{\rm loc}_{S_m \setminus S_n}}\Big)\Big(\sum_{q\in\Delta_{S_m}(S_n)}{}' \Psi(q)\Big) \qquad \text{and}$$

$$\Delta_T(R) \;:=\; \{Z \subset T \,\big|\, Z\cap R \neq \emptyset \neq Z \cap (T\backslash R)\},$$

and the prime on the sum indicates that it is restricted by requiring q to be a link or plaquette.

Thus, we get

$$\left\| \tau_t^{S_m}(A) - \tau_t^{S_n}(A) \right\| \leq \int_{t-}^{t^+} \left\| \left[N(s), \alpha_{s-t}^{S_n}(A(t)) \right] \right\| ds$$

where $t^- = \min\{0, t\}$ and $t^+ = \max\{0, t\}$.

We will now estimate $\left\| \left[N(s), \alpha_{s-t}^{S_n}(A(t)) \right] \right\|$ (Lieb-Robinson estimates).

Define the auxiliary function

$$f(t) := \left[D, \alpha_t^{S_n}(A) \right] \in \mathcal{B}_{S_m},$$

$A \in \mathcal{B}_R$, $R \subset S_n$, and $D \in \mathcal{B}_{S_m}$ is any element with support in $S_m \backslash (S_n)_0 :=$ all sites & links between them in S_m, obtained from either links in $\Lambda^1_{S_m} \backslash \Lambda^1_{S_n}$ or plaquettes in $\Lambda^2_{S_m} \backslash \Lambda^2_{S_n}$.

Then $\|f(0\| \leq 2\|A\|\|D\|\delta_R^{S_n}$ where $\delta_R^{S_n} = 0$ if $R \cap S_m \backslash (S_n)_0 = \emptyset$ (hence $\left[D, A \right] = 0$) and one otherwise.

By strong operator differentiating, we obtain for all $\psi \in \mathcal{H}$ that

$$\frac{d}{dt} f(t)\psi \quad = \quad i\big[\alpha_t^{S_n}(\widetilde{H}_R^{\rm int}), f(t)\big]\psi - i\Big[\alpha_t^{S_n}(A), \big[D, \alpha_t^{S_n}(\widetilde{H}_R^{\rm int})\big]\Big]\psi$$

$$\text{where} \qquad \widetilde{H}_R^{\rm int} \quad := \quad {\sum_{q\in\Delta_{S_n}(R)}}' \Psi(q).$$

Lemma

Let $A, \; B : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ be strong operator continuous maps, such that $A(t)^* = A(t)$ and $\|A(t)\| < M$ for all t for a fixed M, and assume that $\mathcal{H}$ is separable. Then for any $t_0 \in \mathbb{R}$ and $f_0 \in \mathcal{B}(\mathcal{H})$, the ODE

$$\frac{d}{dt} f(t)\psi = i[f(t), A(t)]\psi + B(t)\psi \quad \forall\, \psi \in \mathcal{H}, \quad \text{and} \quad f(t_0) = f_0 \in \mathcal{B}(\mathcal{H}).$$

has a unique solution, and it satisfies $\forall\, t \in \mathbb{R}$:

$$\|f(t)\| \leq \|f(t_0)\| + \int_{t_-}^{t_+} \|B(s)\| ds \qquad t_- := \min\{t_0, t\}, \quad t_+ := \max\{t_0, t\}.$$

Thus

$$\begin{aligned} \|f(t\| &\leq \|f(0)\| + 2\|A\| \int_{t^-}^{t^+} \left\| \left[D, \alpha_r^{S_n}(\widetilde{H}_R^{\mathrm{int}}) \right] \right\| \, dr, \\ \frac{\left\| \left[D, \alpha_t^{S_n}(A) \right] \right\|}{2\|A\|} &\leq \|D\| \delta_R^{S_n} + \sum_{q \in \Delta_{S_n}(R)}{}' \int_{t^-}^{t^+} \left\| \left[D, \alpha_r^{S_n}(\Psi(q)) \right] \right\| \, dr. \end{aligned}$$

As $\Psi(q) \in \mathcal{B}_q$, this inequality can now be iterated . The iteration converges, and produces

$$\frac{\left\| \left[D, \alpha_t^{S_n}(A) \right] \right\|}{2\|A\|} \leq \|D\| \left(\delta_R^{S_n} + \sum_{k=1}^{\infty} \frac{(2\|\Psi\| |t|)^k}{k!} a_k \right),$$

where

$$a_k := \sum_{q_1 \in \Delta_{S_n}(R)}{}' \sum_{q_2 \in \Delta_{S_n}(q_1)}{}' \cdots \sum_{q_k \in \Delta_{S_n}(q_{k-1})}{}' \delta_{q_k}^{S_n}.$$

To estimate the a_k we need to fix the geometry.
Assume S_n is the lattice cube with corner vertices $(\pm n, \pm n, \pm n)$ and $R = S_d$ for d fixed. Then

$$\left|\Delta_{S_n}(R) \cap (\Lambda^1 \cup \Lambda^2)\right| = \sum_{q \in \Delta_{S_n}(R)}{}^{'} 1 \leq 30(2d+1)^2 \geq a_1 = \sum_{q \in \Delta_{S_n}(R)}{}^{'} \delta_q^{S_n}.$$

For the a_k, observe that the sequence

$$(q_1, q_2, \ldots, q_k) \quad \text{with} \quad q_i \in \Delta_{S_n}(q_{i-1})$$

specifies a continuous path where the steps are either links or plaquettes, starting from a $q_1 \in \Delta_{S_n}(R)$ which has a point in R.
As $\Delta_{S_n}(S_r) \cap (\Lambda^1 \cup \Lambda^2) \subset S_{r+1}$ and $\delta_{S_r}^{S_n} = 0$ if $r < n-2$ we conclude that $\delta_{q_k}^{S_n} = 0$ whenever $k < n-d-2$.Thus

$$a_k = \sum_{q_1 \in \Delta_{S_n}(R)}{}^{'} \sum_{q_2 \in \Delta_{S_n}(q_1)}{}^{'} \cdots \sum_{q_k \in \Delta_{S_n}(q_{k-1})}{}^{'} \delta_{q_k}^{S_n} \leq 30(2d+1)^2(48)^{k-1}$$

if $k \geq n-d-2$, and $a_k = 0$ otherwise.

Thus

$$\frac{\left\|\left[D,\alpha_t^{S_n}(A)\right]\right\|}{2\|A\|} \leq \|D\|\Big(\delta_{S_d}^{S_n} + \sum_{k=1}^{\infty}\frac{(2\|\Psi\||t|)^k}{k!}a_k\Big)$$

$$\leq \quad \|D\|(2d+1)^2 \sum_{k=n-d-2}^{\infty}\frac{5(96\|\Psi\||t|)^k}{8(k!)} \qquad \text{if } n > d+4.$$

These are the desired estimates, which we can now apply to Integral estimate
If $n > d+4$ then

$$\left\|\left[\widetilde{N}(s),\alpha_{s-t}^{S_n}(A(t))\right]\right\| \leq 2\|A\|\|\Psi\|30(2n+1)^2(2d+1)^2 \sum_{k=n-d-2}^{\infty}\frac{5(96\|\Psi\||t|)^k}{8(k!)}$$

This gives for $n > d + 4$:

$$\begin{aligned} &\left\|\tau_t^{S_m}(A) - \tau_t^{S_n}(A)\right\| \\ \leq \quad & \frac{75}{192}\|A\|(2d+1)^2(2n+1)^2\frac{(96\|\Psi\||t|)^{n-d-1}}{(n-d-2)!}\exp\left(96\|\Psi\||t|\right). \end{aligned}$$

It is clear that this converges to zero as $n \to \infty$ for any t.
This concludes the proof.

For more information, consult:-

http://arxiv.org/abs/1512.06319

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Thank you!