Tensor-stable Positive Maps

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1 Introduction

Definition 1.1. A linear map $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ is *n-tensor-stable positive* for $n \in \mathbb{N}$ if the map $\mathcal{P}^{\otimes n} : \mathcal{M}_{d_1^n} \to \mathcal{M}_{d_2^n}$ is positive. A linear map $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ is *tensor-stable positive* if $\mathcal{P}$ is n-tensor-stable for all $n \in \mathbb{N}$.

All completely positive maps and completely co-positive maps are tensor-stable positive. We call these maps *trivial tensor-stable positive maps.* We are interested in the existence of the non-trivial tensor-stable postive maps.

2 Partial Answers

Theorem 2.1. *For any $n \in \mathbb{N}$ and $d_1, d_2 \geq 2$, there exists a non-trivial n-tensor-stable postive map $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$.*

sketch of proof. We need a lemma: for a separable matrix $P \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2}$, define

$$\mu_P = \min\{\, \langle \xi | P | \xi \rangle : \xi = \phi \otimes \psi \text{ where } \phi \in \mathbb{C}^{d_1},\ \psi \in \mathbb{C}^{d_2}, \text{ and } \|\phi\| = \|\psi\| = 1\}. \tag{1}$$

Then

$$\min\{\, \langle \Xi | P^{\otimes n} | \Xi \rangle : \Xi = \Phi \otimes \Psi \text{ where } \Phi \in \left(\mathbb{C}^{d_1}\right)^{\otimes n},\ \Psi \in \left(\mathbb{C}^{d_2}\right)^{\otimes}, \text{ and } \|\Phi\| = \|\Psi\| = 1\} = \mu_P^n \tag{2}$$

for all $n \in \mathbb{N}$.

Define

$$P := (|1\rangle\,|1\rangle + |2\rangle\,|2\rangle)\,(\langle 1|\,\langle 1| + \langle 2|\,\langle 2|) + |1\rangle\langle 1| \otimes |2\rangle\langle 2| + |2\rangle\langle 2| \otimes |1\rangle\langle 1| + \sum_{i>2 \text{ or } j>2} |i\rangle\langle i| \otimes |j\rangle\langle j| \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2}. \tag{3}$$

Then P is separable and $\mu_P = \frac{1}{2}$. From the lemma, we have

$$\langle \Psi \otimes \Phi | (P - \epsilon \mathbb{1}_{d_1} \mathbb{1}_{d_2})^{\otimes n} | \Psi \otimes \Phi \rangle \geq 0 \tag{4}$$

for any $\epsilon \in \left[0, \sqrt[n]{2^n + (\frac{1}{2})^n} - 2\right]$. Choose any $\epsilon \in \left(0, \sqrt[n]{2^n + (\frac{1}{2})^n} - 2\right]$. Then the corresponding super operator $\mathcal{P}_\epsilon^{\otimes n}$ is positive, i.e., $\mathcal{P}_\epsilon$ is n-tensor-stable positive. Since P and its partial transpose are not full-rank, they are not positive. This implies that $\mathcal{P}_\epsilon$ is neither completely positive nor completely co-positive. □

We can reduce the existence problem of a non-trivial tensor-stable positve map by restricting the candidates.

Theorem 2.2. *Let $d_1, d_2 \in \mathbb{N}$, $d \in \{d_1, d_2\}$, and for $p \in [-1, 1]$, let*

$$\mathcal{P}_p := \mathcal{W}_p \otimes (\vartheta_d \circ \mathcal{W}_p) : \mathcal{M}_d \otimes \mathcal{M}_d \to \mathcal{M}_d \otimes \mathcal{M}_d, \tag{5}$$

$$\mathcal{W}_p : \mathcal{M}_d \to \mathcal{M}_d, \mathcal{W}_p(X) := \frac{1}{d^2 - 1}\left((d-p)\operatorname{Tr}(X)\mathbb{1}_d - (1 - dp)X^T\right) \text{ for } X \in \mathcal{M}_d. \tag{6}$$

If there exists a non-trivial tensor-stable map $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$, then $\mathcal{P}_p$ is tensor-stable positive for some $p \in [-1, 0)$. If $\mathcal{P}_p$ is tensor-stable positive for some $p \in [-1, 0)$, then it is non-trivial tensor-stable positive.

3 Implications

3.1 The existence of an NPPT bound-entangled state

Theorem 3.1. *If there exists a non-trivial tensor-stable positive map $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$, then there exist NPPT bound-entangled states in $\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_1}$ and $\mathcal{M}_{d_2} \otimes \mathcal{M}_{d_2}$.*

3.2 New bounds on the quantum capacity

Definition 3.2. $R \in \mathbb{R}^+$ is called an *achievable rate* of a quantum channel $\mathcal{T} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ if there exist sequences $(n_\nu)_{\nu\in\mathbb{N}}, (m_\nu)_{\nu\in\mathbb{N}}$ such that $R = \limsup_{\nu\to\infty} \frac{n_\nu \log_2(d)}{m_\nu}$ and

$$\inf_{\mathcal{E},\mathcal{D}} \frac{1}{2}\left\|\mathrm{id}_d^{\otimes n_\nu} - \mathcal{D} \circ \mathcal{T}^{\otimes m_\nu} \circ \mathcal{E}\right\|_\diamond \to 0 \quad \text{as } \nu \to \infty \tag{7}$$

where the infimum is over all quantum channels $\mathcal{E} : \mathcal{M}_d^{\otimes n_\nu} \to \mathcal{M}_{d_1}^{\otimes m_\nu}$ and $\mathcal{D} : \mathcal{M}_d^{\otimes m_\nu} \to \mathcal{M}_{d_2}^{\otimes n_\nu}$. The *quantum capacity* of a quantum channel $\mathcal{T} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ is defined as

$$\mathcal{Q}(\mathcal{T}) := \sup\{R \in \mathbb{R}^+ : R \text{ is an achievable rate.}\} \tag{8}$$

It was known that for any quantum channel $\mathcal{T} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$, we have a *transposition bound* [Holevo, Werner, 2001]

$$\mathcal{Q}(\mathcal{T}) \leq \log_2(\|\vartheta_{d_2} \circ \mathcal{T}\|_\diamond). \tag{9}$$

This bound can be generalized if there exists a surjective, unital and tensor-stable positive map, but not a completely co-positive map.

Theorem 3.3. *Let $\mathcal{T} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ be a quantum channel and $\mathcal{P} : \mathcal{M}_{d_3} \to \mathcal{M}_{d_2}$ be a surjective, unital and tensor-stable positive map. Let $\mathcal{P}^{-1}$ be any right inverse of $\mathcal{P}$. Then*

$$\mathcal{Q}(\mathcal{T}) \leq \frac{\log_2(\|\mathcal{P}^{-1} \circ \mathcal{T}\|_\diamond \|\mathcal{P}^*(\mathbb{1}_{d_2})\|_\infty) \log_2(d_2)}{\log_2(\|\mathcal{P}^*\|_\diamond)}. \tag{10}$$

Note that $\|\vartheta_d^*\|_\diamond = \|\vartheta_d\|_\diamond$ and

$$\frac{\|(\mathrm{id}_d \otimes \vartheta_d)\omega_d\|_1}{\|\omega_d\|_1} = \frac{\left\|\frac{1}{d}\mathbb{F}_d\right\|_1}{\|\omega_d\|_1} = d, \tag{11}$$

so we get

$$\|\vartheta_d\|_\diamond \geq d. \tag{12}$$

By plugging $\mathcal{P} = \vartheta_{d_2}$ and (12) to Theorem 3.3, we can get the transposition bound.

References

[1] Alexander Müller-Hermes, David Reeb, and Michael M. Wolf. Positivity of linear maps under tensor powers. *Journal of Mathematical Physics*, 57(1):015202, 2016.

[2] A. S. Holevo and R. F. Werner. Evaluating capacities of bosonic gaussian channels. *Physical Review A*, 63(3):032312, 2001.